Tamás Keleti, Eötvös Loránd University, Department of Analysis, Budapest,
Múzeum krt. 6-8, H-1088, Hungary; e-mail: elek@konig.elte.hu

A PECULIAR SET IN THE PLANE CONSTRUCTED BY VITUŠKIN, IVANOV AND MELNIKOV

ABSTRACT. In 1963 Vituškin, Ivanov and Melnikov constructed a compact set in the plane with positive linear measure which cannot be mapped, using a contraction onto a segment. Their paper is very concise and obscure. The question of the existence of such a set arose recently again. For this reason in this paper we construct this set again with a detailed proof.

Introduction

Kolmogoroff [3] (in 1932) calls a μ non-negative σ-additive set function on the Borel subsets of $\mathbf{R}^n$ a *measure function*, if a contraction cannot increase the μ-measure and there exists a measure unit J with $\mu(J)=1$. If this measure unit is the k-dimensional unit cube, then μ is said to be a *k-dimensional measure function*.

Kolmogoroff proved that there exist two k-dimensional measure functions: the *minimal* and the *maximal k-dimensional measure* ($\underline{\mu}^k$ and $\bar{\mu}^k$) between which all the other measure functions are included. He also proved that

$$(\diamond) \qquad \underline{\mu}^k(E) = \sup\Big\{\sum_{i=1}^{\infty} \lambda_k(f_i(E_i)) : E_i \subset E \text{ disjoint Borels}, f_i : E_i \to \mathbf{R}^k \text{contraction}\Big\}$$

(where λ_k is the k-dimensional Lebesgue measure.)

It is almost obvious that the k-dimensional Hausdorff measure (μ^k) is a k-dimensional measure function, so $\mu^k \geq \underline{\mu}^k$. Kolmogoroff conjectured that these two measures are equal. Besicovitch [1] disproved this conjecture in 1936 constructing a compact set in the plane with Hausdorff linear measure 2 and minimal linear measure at most $\frac{\sqrt{50}}{4}$. Vituskin, Ivanov and Melnikov [6] proved in 1963 much more: They showed that these two measures are incommensurable (if k=1). They constructed a compact set in the plane with positive Hausdorff linear measure and 0 minimal linear measure.

Recently this construction became interesting again:

A few years ago Miklós Laczkovich asked the following question: Can every measurable subset of $\mathbf{R}^n$ with positive (Lebesgue) measure be mapped, using a contraction onto a (n-dimensional) ball?

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If n=1 then the answer is almost obviously "yes". David Preiss [5] proved that this is also true for $n = 2$. (His result has not been published yet. Jiří Matoušek [4] gave a shorter proof.) But for $n > 2$ the problem is still unsolved and seems to be very hard.

Later the following natural generalization of this problem arose:
Can every (k-dimensional Hausdorff) measurable subset of $\mathbf{R}^n$ with positive k-dimensional Hausdorff measure be mapped, using a contraction onto a (k-dimensional) ball?

This question proved to be very difficult even in the case $k = 1$. David Fremlin conjectured that any compact metric space with positive linear measure can be mapped, using a contraction onto a segment. This would imply that in the case $k = 1$ the answer is also "yes". But Sergey Konyagin disproved Fremlin's conjecture. He did not publish this result because soon after he found the above mentioned paper of A. G. Vituškin, L. D. Ivanov and M. S. Melnikov and he realized that their construction is a counter-example in the plane. Indeed, according to ($\diamond$)

$$\underline{\mu}^1(E) = 0 \Longleftrightarrow \lambda_1(f(E)) = 0 \text{ for any } f : E \to \mathbf{R} \text{ contraction} \Longleftrightarrow$$

$$\Longleftrightarrow E \text{ cannot be mapped, using a contraction onto a segment}$$

This counter-example gives a complete answer for the case $k = 1$. It implies that if $k = 1$ and $n \geq 2$ then the answer for the generalized question is surprisingly "no".

But the paper of Vituškin, Ivanov and Melnikov is very concise and inaccurate. It contains only the sketch of the proofs. Many proofs are omitted, some have errors. Our purpose in this paper is to make the proof correct, complete and understandable. Our proof differs from the original one in some parts, other parts have simply been corrected and explained in greater detail. The construction of the set also slightly differs from the original construction.

Therefore in this paper, using the sketch of Vituškin, Ivanov and Melnikov we prove the following:

Main Theorem. *There exists a compact set in the plane with positive linear measure that cannot be mapped, using a contraction onto a segment.*

Notation

We call a mapping *contraction* if the distance between any two points of the domain is bigger than the distance of their image. (In this paper we could also allow equality, that is we can replace all "contraction" by "Lipschitz-1". In fact in the papers of Kolmogoroff and Besicovitch they allowed equality in the definition of contraction but it is easy to check that this does not make any change.)

We denote the 1-dimensional (linear) Lebesgue and Hausdorff measure by λ and μ respectively. (The linear Lebesgue measure is defined on $\mathbf{R}$, the linear Hausdorff measure is defined on $\mathbf{R}^n$ for arbitrary n.) The *girth* of the set A ($c(A)$) is defined by

$$c(A) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam} A_n \ : \ A \subset \cup_{n=1}^{\infty} A_n \right\}.$$

Obviously $\mu(A) \geq c(A)$. It is easy to verify that if A is compact then for computing c(A) it is enough to consider the finite covers of A.

Denote the closed neighborhood of a set H with radius δ by $U_\delta(H)$, that is

$$U_\delta(H) = \{x \ : \ \mathrm{dist}(x, H) \leq \delta\}.$$

Denote the *Hausdorff distance* of two non-empty compact sets A and B by $d_H(A, B)$, that is

$$d_H(A, B) = \inf\{\delta \geq 0 \ : \ A \subset U_\delta(B), \ B \subset U_\delta(A)\}.$$

The construction and the proof

1. Let R be a horizontal segment (with length d), k and m integers greater than 1. Divide this segment into $2k$ equal pieces and put every second piece above the previous one by the $2m$-th part of the length of the small segments (see Figure 1.). Denote this transformation by $P_{k,m}$. Let $P_{k,m}(R, 0)$ be the union of the lower segments, $P_{k,m}(R, 1)$ the union of the upper segments, that is if

$$R = \{(x, y) : a \leq x \leq a + d, y = c\}$$

then

$$P_{k,m}(R, 0) = \bigcup_{i=0}^{k-1} \{(x, y) : a + \frac{2id}{2k} \leq x \leq a + \frac{(2i+1)d}{2k}, y = c\}$$

$$P_{k,m}(R, 1) = \bigcup_{i=0}^{k-1} \{(x, y) : a + \frac{2id}{2k} \leq x \leq a + \frac{(2i+1)d}{2k}, y = c + \frac{d}{4km}\}$$

$$P_{k,m}(R) = P_{k,m}(R, 0) \cup P_{k,m}(R, 1).$$

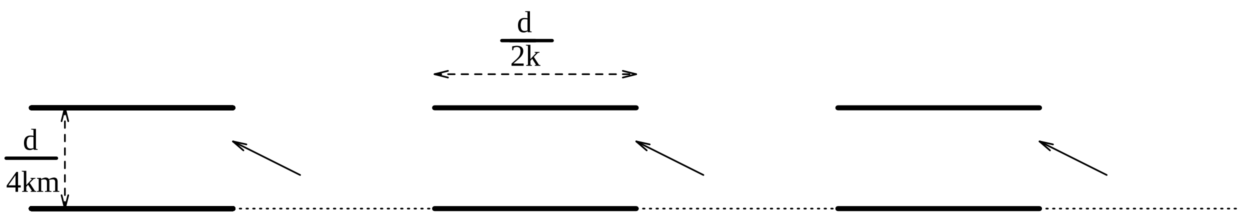

Figure 1. Transformation $P_{k,m}$ (k=3)

We can apply the transformation $P_{k,m}$ to a finite system of horizontal segments applying $P_{k,m}$ for each segment, that is if

$$S = \{r_1, r_2, \ldots, r_m\}$$

then let

$$P_{k,m}(S, \tau) = \bigcup_{p=1}^{m} P_{k,m}(r_p, \tau), \qquad P_{k,m}(S) = \bigcup_{p=1}^{m} P_{k,m}(r_p).$$

Let M be a horizontal segment with length d_0, let $m_0, m_1, m_2, \dots$ be a nondecreasing sequence of integers greater than 1, let $k_0, k_1(0), k_1(1), \dots, k_n(0), k_n(1), \dots$ be integers greater than 1. We will choose these numbers later properly. We are converting M by transformations $P_{k,m}$ in the following way:

First we apply P_{k_0,m_0} to M (see Figure 2.). Let

$$M(0) = P_{k_0,m_0}(M, 0) \quad M(1) = P_{k_0,m_0}(M, 1).$$

At the second step we apply $P_{k_1(0),m_1}$ to $M(0)$ and $P_{k_1(1),m_1}$ to $M(1)$. Let

$$M(\tau_1, \tau_2) = P_{k_1(\tau_1),m_1}(M(\tau_1), \tau_2).$$

Continuing this we apply $P_{k_j(\tau_j),m_j}$ to $M(\tau_1, \dots, \tau_j)$ at the $j+1$-st step and let

$$M(\tau_1, \dots, \tau_j, \tau_{j+1}) = P_{k_j(\tau_j),m_j}(M(\tau_1, \dots, \tau_j), \tau_{j+1}).$$

Let

$$M^j = \bigcup_{\tau_1,\dots,\tau_j=0,1} M(\tau_1, \dots, \tau_j).$$

The set we want to construct will be the limit of the sets M^j by the Hausdorff metric.

By construction the set $M(\tau_1, \dots, \tau_j)$ consists of $k_0 k_1(\tau_1) \dots k_{j-1}(\tau_{j-1})$ horizontal segments lying on the same height. Denote them by $r_1(\tau_1, \dots, \tau_j), r_2(\tau_1, \dots, \tau_j), \dots$ from left to right (see Figure 2.). Denoting their length by $l(\tau_1, \dots, \tau_j)$ we have

$$l(\tau_1, \dots, \tau_j) = \mu(r_p(\tau_1, \dots, \tau_j)) = \frac{d_0}{2^j k_0 k_1(\tau_1) \dots k_{j-1}(\tau_{j-1})}. \tag{1}$$

Denote the distance of the horizontal lines of $M(\tau_1, \dots, \tau_j, 0)$ and $M(\tau_1, \dots, \tau_j, 1)$ by $\delta(\tau_1, \dots, \tau_j)$. Then by the definition of $P_{k,m}$

$$\delta(\tau_1, \dots, \tau_j) = \frac{l(\tau_1, \dots, \tau_j, \tau_{j+1})}{2m_j} = \frac{d_0}{m_j 2^{j+2} k_0 k_1(\tau_1) \dots k_j(\tau_j)}. \tag{2}$$

Since the sequence (m_j) is nondecreasing and each $k_j(\tau)$ is at least 2, using (2) we obtain that

$$\delta(\tau_1, \dots, \tau_j, \tau_{j+1}) \leq \frac{\delta(\tau_1, \dots, \tau_j)}{4}. \tag{3}$$

This implies that the levels of the sets $M(\tau_1, \dots, \tau_j)$ do not 'cross each other', that is $M(\tau_1, \dots, \tau_j)$ lies higher than $M(\tau_1', \dots \tau_j')$, if and only if in the binary system $0, \tau_1 \dots \tau_j$ is greater than $0, \tau_1' \dots \tau_j'$.

Let

$$M(\tau_1, \dots, \tau_j)^i = \bigcup_{\tau_{j+1},\dots,\tau_{j+i}=0,1} M(\tau_1, \dots, \tau_j, \tau_{j+1}, \dots, \tau_{j+i}).$$

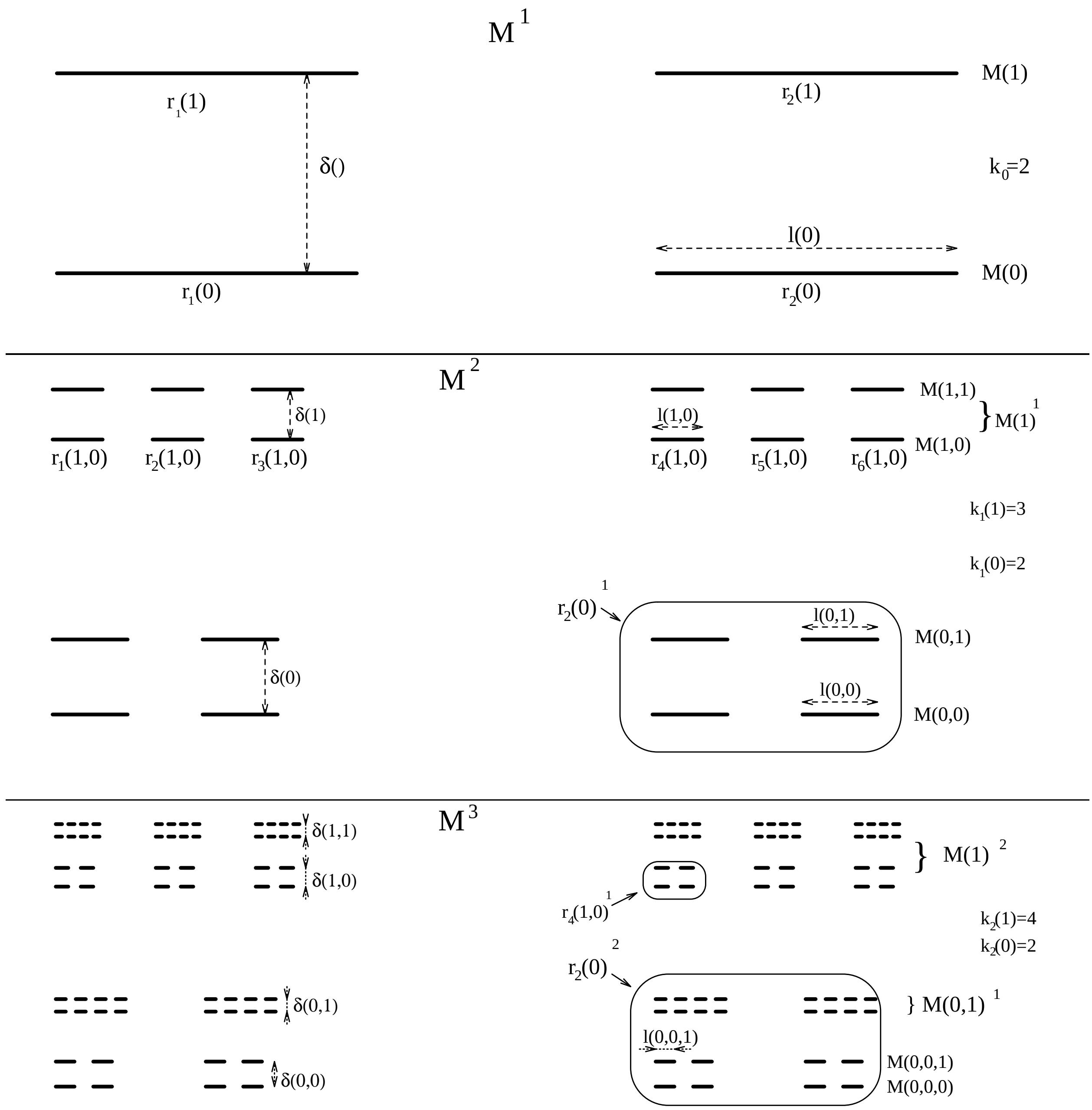

Figure 2.

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Let $r_p(\tau_1, \ldots, \tau_j)^i$ be the part of the set $M(\tau_1, \ldots, \tau_j)^i$ which originated from the set $r_p(\tau_1, \ldots, \tau_j)$ (see Figure 2.). Let $J_i(\tau_1, \ldots, \tau_j)$ be the number of the segments in $r_p(\tau_1, \ldots, \tau_j)^i$. By construction

(4) $$J_i(\tau_1, \ldots, \tau_j) = 2^i \sum_{\tau_{j+1}, \ldots, \tau_{j+i-1} = 0,1} k_j(\tau_j) \ldots k_{j+i-1}(\tau_{j+i-1}).$$

Proposition 1.1 *The sequence $M^1, M^2, \ldots$ is convergent by the Hausdorff metric.*

Proof. Since $\mathbf{R}^2$ is complete the space of its nonempty compact subsets with the Hausdorff metric is also complete. Therefore it is enough to prove that $M^1, M^2, \dots$ is a Cauchy sequence.

By the definition of $P_{k,m}$

$$P_{k,m}(R) \subset U_{\frac{d}{4km}}(R) \quad \text{and} \quad R \subset U_{\frac{d}{2k}}(P_{k,m}(R)),$$

where R is a horizontal segment with length d. Using this, by construction

$$M(\tau_1, \dots, \tau_j, \tau_{j+1}) \subset U_{\delta(\tau_1,\dots,\tau_j)}(M(\tau_1, \dots, \tau_j)) \qquad (\tau_1, \dots, \tau_{j+1} = 0 \text{ or } 1),$$

so

$$M^{j+1} \subset U_{\max \delta(\tau_1,\dots,\tau_j)}(M^j).$$

On the other hand

$$M(\tau_1, \dots, \tau_j) \subset U_{l(\tau_1,\dots,\tau_j,\tau_{j+1})}(M(\tau_1, \dots, \tau_j, \tau_{j+1})), \text{ so } M^j \subset U_{\max l(\tau_1,\dots,\tau_j,\tau_{j+1})}(M^{j+1}),$$

therefore

$$d_H(M^j, M^{j+1}) \leq \max\big(\max \delta(\tau_1, \dots, \tau_j), \max l(\tau_1, \dots, \tau_j, \tau_{j+1})\big).$$

By construction

$$\delta(\tau_1, \dots, \tau_j) = \frac{l(\tau_1, \dots, \tau_j, \tau_{j+1})}{2m_j} \leq l(\tau_1, \dots, \tau_j, \tau_{j+1}).$$

On the other hand the length of the segments (the numbers $l(\dots)$) are always divided at least by 2, so

$$l(\tau_1, \dots, \tau_j, \tau_{j+1}) \leq \frac{d_0}{2^{j+1}} \qquad (\text{where } d_0 \text{ is the length of } M).$$

Comparing this to the previous inequalities we get

$$d_H(M^j, M^{j+1}) \leq \frac{d_0}{2^{j+1}},$$

which implies that $M^1, M^2, \dots$ is a Cauchy sequence indeed. □

Definition 1.2 *Let* $M^* = \lim_{j\to\infty} M^j$.

2. Now our aim is to prove, that if

$$(*) \qquad k_j(\tau) \geq m_{j-1} \text{ for all } j \text{ and } \tau$$

then $\mu(M^*) \geq \frac{d_0}{20}$. In fact we prove that $c(M^*) \geq \frac{d_0}{20}$.

Proposition 2.1 *The condition* $(*)$ *implies that*

$$l(\tau_1, \ldots, \tau_j, \tau_{j+1}, \tau_{j+2}) \leq \delta(\tau_1, \ldots, \tau_j)$$

for arbitrary τ *parameters.*

Proof. Using the formulas of l and δ ((1) and (2)) we get

$$l(\tau_1, \ldots, \tau_j, \tau_{j+1}, \tau_{j+2}) = \frac{m_j}{k_{j+1}(\tau_{j+1})} \delta(\tau_1, \ldots, \tau_j)$$

which implies the statement if $(*)$ holds. □

Proposition 2.2 *Let* Q *be an axis-parallel square with side* a. *Then* $(*)$ *implies that* $\mu(Q \cap M^n) \leq 20a$ *for arbitrary* n.

Proof. Let j be the maximal index for which there exist $\tau_1, \ldots, \tau_j$ such that

$$Q \cap M^n \subset M(\tau_1, \ldots, \tau_j)^{n-j}.$$

(There exists such a j since $Q \cap M^n \subset M(\emptyset)^n$ and $j \leq n$.)

If $n - j = 0$ or 1, then it is easy to prove the statement: In this case $M(\tau_1, \ldots, \tau_j)^{n-j}$ lies on 1 or 2 horizontal lines, so $Q \cap M^n$ does so as well. On the other hand $Q \cap M^n$ is covered by an axis-parallel square with side a, so $\mu(Q \cap M^n) \leq 2a$. Therefore we can assume that $n - j \geq 2$.

Since j is maximal and

$$M(\tau_1, \ldots, \tau_j)^{n-j} = M(\tau_1, \ldots, \tau_j, 0)^{n-j-1} \cup M(\tau_1, \ldots, \tau_j, 1)^{n-j-1},$$

we have

$$Q \cap M(\tau_1, \ldots, \tau_j, 0)^{n-j-1} \neq \emptyset \quad \text{and} \quad Q \cap M(\tau_1, \ldots, \tau_j, 1)^{n-j-1} \neq \emptyset.$$

The points of $M(\tau_1, \ldots, \tau_j, 0)^{n-j-1}$ lies by at most

$$\delta(\tau_1, \ldots, \tau_j, 0) + \delta(\tau_1, \ldots, \tau_j, 0, 1) + \delta(\tau_1, \ldots, \tau_j, 0, 1, 1) + \ldots \leq \frac{1}{2}\delta(\tau_1, \ldots, \tau_j)$$

higher than the line of $M(\tau_1, \ldots, \tau_j)$. (We used the inequality (3).)

On the other hand all the points of $M(\tau_1, \ldots, \tau_j, 1)^{n-j-1}$ lie not lower than the line of $M(\tau_1, \ldots, \tau_j, 1)$, so they lie by at least $\delta(\tau_1, \ldots, \tau_j)$ higher then the line of $M(\tau_1, \ldots, \tau_j)$. Therefore if both $M(\tau_1, \ldots, \tau_j, 0)^{n-j-1}$ and $M(\tau_1, \ldots, \tau_j, 1)^{n-j-1}$ intersect the square Q with height a then

$$\frac{1}{2}\delta(\tau_1, \ldots, \tau_j) \leq a.$$

Using this and Proposition 2.1 we get

$$l(\tau_1, \ldots, \tau_j, \tau, \eta) \leq 2a \qquad (\tau, \eta = 0, 1).$$

On the other hand for fixed τ and η the segments of $M(\tau_1,\ldots,\tau_j,\tau,\eta)$ that intersect the vertical strip of Q are covered by the surrounding vertical strip with width $l(\tau_1,\ldots,\tau_j,\tau,\eta)+a+l(\tau_1,\ldots,\tau_j,\tau,\eta)$. Therefore using the previous inequality these segments lie in a vertical strip with width $5a$. Since (for fixed τ and η) they lie in a horizontal line we obtain that the sum of the lengths of these segments is at most $5a$. The set $M(\tau_1,\ldots,\tau_j)^2$ consists of 4 set of type $M(\tau_1,\ldots,\tau_j,\tau,\eta)$ therefore the sum of the lengths of those segments of $M(\tau_1,\ldots,\tau_j)^2$ that intersect the vertical strip of Q is at most $20a$.

By construction only those segments of $M(\tau_1,\ldots,\tau_j)^{n-j} = (M(\tau_1,\ldots,\tau_j)^2)^{n-j-2}$ can intersect the vertical strip of Q that originated of the segments above. (We used that $r_p(\tau_1,\ldots,\tau_j,\tau,\eta)^{n-j-2}$ is in the strip above $r_p(\tau_1,\ldots,\tau_j,\tau,\eta)$.) So the sum of those segments of $M(\tau_1,\ldots,\tau_j)^{n-j}$ that intersect the vertical strip of Q is also at most $20a$. On the other hand $Q\cap M^n \subset M(\tau_1,\ldots,\tau_j)^{n-j}$, so $Q\cap M^n$ is a subset of the union of these subsets, therefore $\mu(Q\cap M^n)\le 20a$. □

Proposition 2.3 *If $(*)$ holds then*

$$c(M^n)\ge \frac{1}{20}d_0.$$

Proof. Let $M^n=\bigcup_1^\infty H_i$. We have to show that $\sum \text{diam} H_i \ge \frac{d_0}{20}$. For a fixed i take an axis-parallel square Q_i with side $\text{diam} H_i$ that cover H_i. Then $H_i\subset Q_i\cap M^n$, so using the previous proposition we get $\mu(H_i)\le\mu(Q_i\cap M^n)\le 20\text{diam}H_i$, that is $\text{diam}H_i\ge\frac{\mu(H_i)}{20}$. Therefore

$$\sum \text{diam}H_i \ge \sum\frac{\mu(H_i)}{20}\ge\frac{\mu(M^n)}{20}=\frac{d_0}{20}. \square$$

Proposition 2.4

$$c(M^*)\ge\frac{d_0}{20},$$

if $(*)$ holds.

Proof. It is an easily provable fact about the Hausdorff metric and the girth of a set that if $\lim_{n\to\infty}A_n = A$ (by Hausdorff metric) then $c(A)\ge\liminf_{n\to\infty}c(A_n)$. Using this for M^n, $(*)$ and the previous proposition we obtain the proposition. □

3. Now we have to find parameters m_j and $k_j(\tau)$ which satisfy $(*)$ and for which M^* cannot be mapped to a segment by contraction. First we need some new notation and some statements without new conditions for the parameters.

The set $r_p(\tau_1,\ldots,\tau_j,\tau)^1$ consists of $2k_{j+1}(\tau)$ small segments. Let us denote those that belong to $M(\tau_1,\ldots,\tau_j,\tau,\eta)$ by $r_{p,1}(\tau_1,\ldots,\tau_j,\tau,\eta),\ldots,r_{p,k_{j+1}(\tau)}(\tau_1,\ldots,\tau_j,\tau,\eta)$ from left to right. Denote by $r_{pq}(\tau_1,\ldots,\tau_j,\tau,\eta)^{n-1}$ the part of the set $M(\tau_1,\ldots,\tau_j,\tau)^n$ which originated from $r_{pq}(\tau_1,\dots,\tau_j,\tau,\eta)$ (see Figure 3.). Then

$$M(\tau_1,\dots,\tau_j,\tau)^n = \bigcup_{p=1}^{k_0k_1(\tau_1)\dots k_j(\tau_j)} r_p(\tau_1,\dots,\tau_j,\tau)^n =$$

$$= \bigcup_{p=1}^{k_0k_1(\tau_1)\dots k_j(\tau_j)} \bigcup_{q=1}^{k_{j+1}(\tau)} r_{pq}(\tau_1,\dots,\tau_j,\tau,0)^{n-1} \cup r_{pq}(\tau_1,\dots,\tau_j,\tau,1)^{n-1}.$$

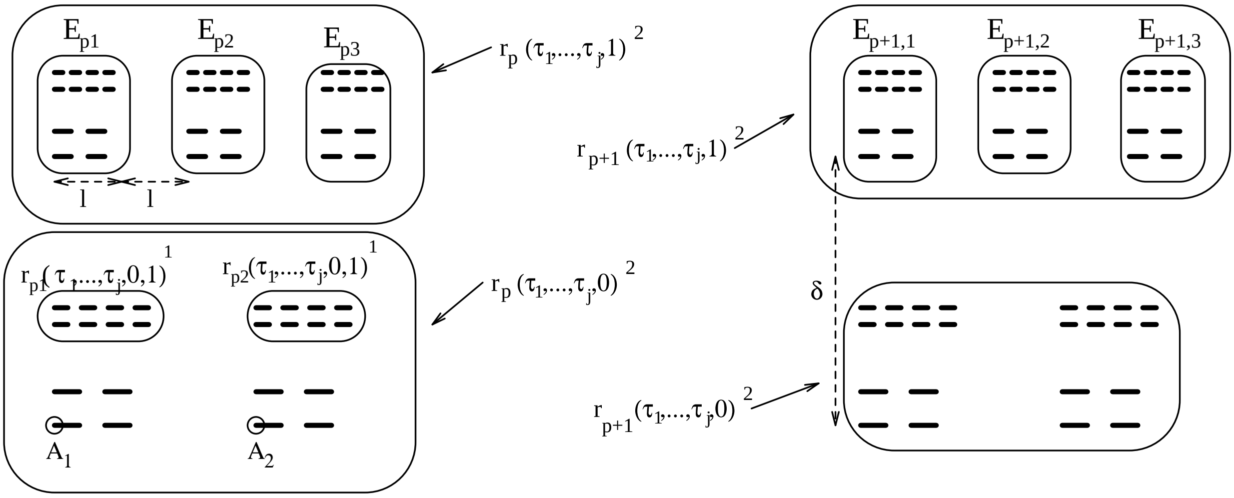

Figure 3. A piece of $M(\tau_1,\dots,\tau_j)^{n+1}$ if $n=2$

Notation 3.1 For given n and $\tau_1,\dots,\tau_j$ let (see Figure 3.)

$$\begin{aligned} E_{pq} &= r_{pq}(\tau_1,\dots,\tau_j,1,0)^{n-1} \cup r_{pq}(\tau_1,\dots,\tau_j,1,1)^{n-1} \\ L &= \mu(M(\tau_1,\dots,\tau_j)^{n+1}) \\ l &= l(\tau_1,\dots,\tau_j,1,0) \\ \delta &= \delta(\tau_1,\dots,\tau_j). \end{aligned}$$

Proposition 3.2 *If $(*)$ holds then*

$$d_H(r_p(\tau_1,\dots,\tau_j,0)^n, r_p(\tau_1,\dots,\tau_j,1)^n) \le 4\delta \quad \big(= 4\delta(\tau_1,\dots,\tau_j)\ \big),$$

where d_H is the Hausdorff metric.

Proof. By A_q we denote the left end-points of the segments $r_{p,q}(\tau_1,\dots,\tau_j,0,0)$ $(q=1,\dots,k_{j+1}(0))$ (see Figure 3.). These points are the left end-points of the parts with length $2l(\tau_1,\dots,\tau_j,0,0)$ of the partition of $r_p(\tau_1,\dots,\tau_j,0)$. So using this and Proposition 2.1 we obtain

$$r_p(\tau_1,\dots,\tau_j,0) \subset U_{2l(\tau_1,\dots,\tau_j,0,0)}(\{A_1,\dots,A_{k_{j+1}(0)}\}) \subset U_{2\delta}(\{A_1,\dots,A_{k_{j+1}(0)}\})$$

The set $r_p(\tau_1,\dots,\tau_j,1)^n$ is contained by the rectangle above $r_p(\tau_1,\dots,\tau_j,0)$ with height

$$\delta(\tau_1,\dots,\tau_j)+\delta(\tau_1,\dots,\tau_j,1)+\delta(\tau_1,\dots,\tau_j,1,1)+\dots \leq 2\delta(\tau_1,\dots,\tau_j)=2\delta.$$

So we can conclude that

$$r_p(\tau_1,\dots,\tau_j,1)^n \subset U_{4\delta}(\{A_1,\dots,A_{k_{j+1}(0)}\}).$$

Since the left end-points of the segments remain in every step, $\{A_1,\dots,A_{k_{j+1}(0)}\} \subset r_p(\tau_1,\dots,\tau_j,0)^n$, so

$$r_p(\tau_1,\dots,\tau_j,1)^n \subset U_{4\delta}(r_p(\tau_1,\dots,\tau_j,0)^n).$$

The set $r_p(\tau_1,\dots,\tau_j,0)^n$ is contained by the rectangle below $r_p(\tau_1,\dots,\tau_j,1)$ with height δ, so similarly as above we obtain

$$r_p(\tau_1,\dots,\tau_j,0)^n \subset U_{3\delta}(r_p(\tau_1,\dots,\tau_j,1)^n).$$

Therefore

$$d_H(r_p(\tau_1,\dots,\tau_j,0)^n, r_p(\tau_1,\dots,\tau_j,1)^n) \leq 4\delta. \ \square$$

Proposition 3.3

$$\text{diam}E_{pq} \leq 2l \quad \left(= 2l(\tau_1,\dots,\tau_j,1,0) \ \right)$$

Proof. The set E_{pq} is contained by the rectangle above $r_{pq}(\tau_1,\dots,\tau_j,1,0)$ with width $l(\tau_1,\dots,\tau_j,1,0)=l$ and height $\delta(\tau_1,\dots,\tau_j,1)+\delta(\tau_1,\dots,\tau_j,1,1)+\delta(\tau_1,\dots,\tau_j,1,1,1)+\dots$. Applying (2) and (3)

$$\delta(\tau_1,\dots,\tau_j,1)+\delta(\tau_1,\dots,\tau_j,1,1)+\dots \leq \delta(\tau_1,\dots,\tau_j,1)(1+\frac{1}{4}+\frac{1}{16}+\dots) \leq$$
$$\leq 2\delta(\tau_1,\dots,\tau_j,1) = 2\frac{l(\tau_1,\dots,\tau_j,1,0)}{2m_{j+1}} = \frac{l}{m_{j+1}} \leq l.$$

So

$$\text{diam}E_{pq} \leq \sqrt{2}l \leq 2l. \ \square$$

Proposition 3.4

$$d_H(E_{pq},E_{p,q+1}) \leq 2l \quad \left(= 2\mu(r_p(\tau_1,\dots,\tau_j,1,0) \ \right)$$

Proof. By construction if we translate E_{pq} horizontally right by $2l$ we get $E_{p,q+1}$, which implies the statement. □

Now we are choosing the parameters in such a way that the set we get after N step (M^N) can be mapped by contraction only onto a very small segment. For this we can make the following restriction: let all the numbers m_j, $k_j(0)$ and k_0 are equal to N. (Since in the first N step we use only the parameters m_j and $k_j(\tau)$ with index smaller than N it is enough to take care of these parameters.) We will need the following condition:

$$(**) \qquad N J_{N-j}(\tau_1, \ldots, \tau_{j-1}, 0) \le k_j(1) \qquad (j = 1, 2, \ldots, N-1).$$

Proposition 3.5 *For a fixed N we can choose the numbers $k_j(1)$ $(j = 1, 2, \ldots, N-1)$ in such a way that $(*)$ and $(**)$ hold if the numbers $k_j(0)$, m_j and k_0 $(j \le N-1)$ are all equal to N.*

Proof. Using (4) and $k_j(0) = N$

$$J_{N-j}(\tau_1, \ldots, \tau_{j-1}, 0) = 2^{N-j} \sum_{\tau_{j+1}, \ldots, \tau_{N-1} = 0,1} k_j(0) k_{j+1}(\tau_{j+1}) \ldots k_{N-1}(\tau_{N-1}) =$$
$$= 2^{N-j} N \sum_{\tau_{j+1}, \ldots, \tau_{N-1} = 0,1} k_{j+1}(\tau_{j+1}) \ldots k_{N-1}(\tau_{N-1}).$$

Therefore $J_{N-j}(\tau_1, \ldots, \tau_{j-1}, 0)$ depends only on the parameters with index greater than j, so we can choose the numbers $k_j(1)$ by recursion in the following way:

Let $k_{N-1}(1) = N$. If $k_{N-1}(1), k_{N-2}(1), \ldots k_{j+1}(1)$ are already defined then let $k_j(1) = N J_{N-j}(\tau_1, \ldots, \tau_{j-1}, 0)$. Then $(**)$ holds with equality. The condition $(*)$ also obviously holds since $k_j(1) \ge N$, $k_j(0) = N$, $m_j = N$ for all j. □

(It is easy to verify that

$$k_j(1) = \frac{(4N)^{2^{N-j}}}{4}$$

is also a good choice.)

Proposition 3.6 *If the numbers k_0, $k_i(0)$, m_i $(i = 1, \ldots, N-1)$ are equal to N, the condition $(**)$ holds and $j + n \le N$, then*

$$\lambda(f(M(\tau_1, \ldots, \tau_j)^n)) \le \frac{20}{n} \mu(M(\tau_1, \ldots, \tau_j)^n),$$

where f is an arbitrary $M((\tau_1, \ldots, \tau_j)^n) \to \mathbf{R}$ contraction. (λ is the 1-dimensional Lebesgue, μ is the 1-dimensional Hausdorff measure.)

As a special case ($j = 0$ and $n = N$), *with the same condition we obtain*

$$\lambda(f(M^N)) \le \frac{20}{N} \mu(M^N) = \frac{20}{N} d_0.$$

Proof. We shall prove the statement by induction on n. Since f is contraction, $\lambda(f(M(\tau_1,\dots,\tau_j)^n)) \le \mu(M(\tau_1,\dots,\tau_j)^n)$, so the theorem is obvious if $n \le 20$.

Suppose that $n \ge 20$ and that the proposition is valid for $1,2,\dots,n$. We have to show that then it is also valid for $n+1$, that is $\lambda(f(M(\tau_1,\dots,\tau_j)^{n+1})) \le \frac{20}{n+1}L$. (For the fixed n and $\tau_1,\dots,\tau_j$ we use the notation E_{pq}, L, l, and δ (Notation 3.1).)

By construction

$$\mu(M(\tau_1,\dots,\tau_j,0)^n) = \mu(M(\tau_1,\dots,\tau_j,1)^n) = \frac{1}{2}\mu(M(\tau_1,\dots,\tau_j)^{n+1}) = \frac{1}{2}L, \tag{5}$$

and

$$M(\tau_1,\dots,\tau_j)^{n+1} = M(\tau_1,\dots,\tau_j,0)^n \cup M(\tau_1,\dots,\tau_j,1)^n. \tag{6}$$

Case I.

$$\lambda(f(M(\tau_1,\dots,\tau_j,0)^n)) \le \frac{9}{10}\frac{20}{n}\mu(M(\tau_1,\dots,\tau_j,0)^n).$$

In this case using (6), the induction assumption, (5) and finally $n \ge 20$:

$$\begin{aligned}\lambda(f(M(\tau_1,\dots,\tau_j)^{n+1})) &\le \lambda(f(M(\tau_1,\dots,\tau_j,0)^n)) + \lambda(f(M(\tau_1,\dots,\tau_j,1)^n)) \le \\ &\le \frac{9}{10}\frac{20}{n}\mu(M(\tau_1,\dots,\tau_j,0)^n) + \frac{20}{n}\mu(M(\tau_1,\dots,\tau_j,1)^n) = \\ &= (\frac{18}{n} + \frac{20}{n})\frac{1}{2}L \le \\ &\le \frac{20}{n+1}L.\end{aligned}$$

Case II.

$$\lambda(f(M(\tau_1,\dots,\tau_j,0)^n)) > \frac{9}{10}\frac{20}{n}\mu(M(\tau_1,\dots,\tau_j,0)^n)\left(=\frac{9}{n}L\right).$$

The proof is quite complicated in this case, so by way of introduction we sketch the motivation of the proof.

According to (6) if we show that a big part of $f(M(\tau_1,\dots,\tau_j,1)^n)$ is covered by $f(M(\tau_1,\dots,\tau_j,0)^n)$, then this and the induction assumption would imply that the set $f(M(\tau_1,\dots,\tau_j)^{n+1})$ is small.

We shall show that for any index p $f(r_p(\tau_1,\dots,\tau_j,0)^n)$ covers quite a big part of $f(r_p(\tau_1,\dots,\tau_j,1)^n)$. For this we shall subdivide the set $f(r_p(\tau_1,\dots,\tau_j,1)^n)$ into the union of the sets $f(r_{pq}(\tau_1,\dots,\tau_j,1,0)^{n-1} \cup r_{pq}(\tau_1,\dots,\tau_j,1,1)^{n-1}) = f(E_{pq})$ and we shall show that among these subsets quite a lot are covered by a type of small segments of $f(r_p(\tau_1,\dots,\tau_j,0)^n)$. It is made possible by $(**)$: This condition implies that in each step we subdivide the upper parts (with last index 1) into much more pieces than the lower ones. For this reason the small segments of $f(r_p(\tau_1,\dots,\tau_j,0)^n)$ are much bigger than the ones of $f(r_p(\tau_1,\dots,\tau_j,1)^n)$ and what is more they are also big relative to

$f(r_{pq}(\tau_1,\ldots,\tau_j,1,0)^{n-1} \cup r_{pq}(\tau_1,\ldots,\tau_j,1,1)^{n-1}) = f(E_{pq})$, which consists small segments of $f(r_p(\tau_1,\ldots,\tau_j,1)^n)$.

Let

$$A_p = \min(f(r_p(\tau_1,\ldots,\tau_j,0)^n)), \quad A'_p = \min(f(r_p(\tau_1,\ldots,\tau_j,1)^n)),$$

$$B_p = \max(f(r_p(\tau_1,\ldots,\tau_j,0)^n)), \quad B'_p = \max(f(r_p(\tau_1,\ldots,\tau_j,1)^n)),$$

$$\big(\text{where } p = 1,2,\ldots,k(\tau_1,\ldots,\tau_j)\big).$$

We shall prove that a significant part of $f(r_p(\tau_1,\ldots,\tau_j,0)^n))$ lies in $A'_pB'_p$: Since f is contraction Proposition 3.2 implies that

$$d_H(f(r_p(\tau_1,\ldots,\tau_j,0)^n), f(r_p(\tau_1,\ldots,\tau_j,1)^n)) \leq 4\delta,$$

so

$$|A'_p - A_p| \leq 4\delta, \qquad |B'_p - B_p| \leq 4\delta.$$

This implies that

$$\lambda(A_pB_p \setminus A'_pB'_p) \leq |A'_p - A_p| + |B'_p - B_p| \leq 8\delta.$$

Using this and $f(r_p(\tau_1,\ldots,\tau_j,0)^n) \subset A_pB_p$ we obtain that

$$\lambda(f(r_p(\tau_1,\ldots,\tau_j,0)^n) \cap A'_pB'_p)) \geq \lambda(f(r_p(\tau_1,\ldots,\tau_j,0)^n)) - 8\delta. \tag{7}$$

Using the decomposition of the set $M(\tau_1,\ldots,\tau_j,1)^n$ we saw earlier (see Figure 3.) we obtain

(8)

$$f(M(\tau_1,\ldots,\tau_j,1)^n) = \bigcup_{p=1}^{k_0k_1(\tau_1)\ldots k_j(\tau_j)} f(r_p(\tau_1,\ldots,\tau_j,1)^n) = \bigcup_{p=1}^{k_0k_1(\tau_1)\ldots k_j(\tau_j)} \bigcup_{q=1}^{k_{j+1}(1)} f(E_{pq}).$$

Fix p. We shall show that the small segments of $f(r_p(\tau_1,\ldots,\tau_j)^n) \cap A'_pB'_p$ cover a lot of sets of type $f(E_{pq})$ $(q = 1,2,\ldots,k_{j+1}(1))$:

Let $A_{pq} = \min f(E_{pq})$. According to Proposition 3.4 the distance between two points of type A_{pq} with adjacent index q is at most $2l$. Using Proposition 3.3 we obtain

$$f(E_{pq}) \subset \big[A_{pq}, \quad A_{pq} + 2l\big]. \tag{9}$$

By the definition of A'_p and B'_p, using (8) and (9) there exists a point of type A_{pq} coinciding with A'_p and there also exists a point of type A_{pq} to the right from $B'_p - 2l$.

Let CD be an arbitrary subsegment of $A'_pB'_p$. Then $[C, D-2l] \subset [A'_p, B'_p - 2l]$, so using the previous observations at least $\frac{(D-2l)-C}{2l} - 1 = \frac{D-C}{2l} - 2$ points of type A_{pq} lie in $[C, D-2l]$. Then from (9) the sets $f(E_{pq})$ corresponding to these indexes q lie in CD. Therefore CD contains at least $\frac{D-C}{2l} - 2$ sets of these type.

By the definition of J_n, $f(r_p(\tau_1,\dots,\tau_j,0)^n)\cap A'_pB'_p$ consists at most $J_n(\tau_1,\dots,\tau_j,0)$ small segments. The sum of their lengths is $\lambda(f(r_p(\tau_1,\dots,\tau_j,0)^n)\cap A'_pB'_p)$, so according to (7) at least $\lambda(f(r_p(\tau_1,\dots,\tau_j,0)^n))-8\delta$. For this reason using the result of the previous paragraph these segments contain at least $\frac{\lambda(f(r_p(\tau_1,\dots,\tau_j,0)^n))-8\delta}{2l}-2J_n(\tau_1,\dots,\tau_j,0)$ sets of type $f(E_{pq})$ (for the fixed p). Therefore $f(r_p(\tau_1,\dots,\tau_j,0)^n)$ itself also covers at least so many sets of this type.

Since this is true for arbitrary $p=1,2,\dots,k_0k_1(\tau_1)\dots k_j(\tau_j)$, denoting by T the number of the sets of type $f(E_{pq})$ covered by the whole $f(M(\tau_1,\dots,\tau_j,0)^n)$,

(10)

$$T\geq\frac{\lambda(f(M(\tau_1,\dots,\tau_j,0)^n))-8\delta k_0k_1(\tau_1)\dots k_j(\tau_j)}{2l}-2k_0k_1(\tau_1)\dots k_j(\tau_j)J_n(\tau_1,\dots,\tau_j,0).$$

Since according to (8) $f(M(\tau_1,\dots,\tau_j,1)^n)$ consists of $k_0k_1(\tau_1)\dots k_j(\tau_j)k_{j+1}(1)$ sets of type $f(E_{pq})$, denoting by K the number of the uncovered sets of this type,

$$K=k_0k_1(\tau_1)\dots k_j(\tau_j)k_{j+1}(1)-T. \tag{11}$$

Using the induction assumption for $r_{pq}(\tau_1,\dots,\tau_j,1,0)^{n-1}$ and $r_{pq}(\tau_1,\dots,\tau_j,1,1)^{n-1}$ we get

$$\lambda(f(r_{pq}(\tau_1,\dots,\tau_j,1,\tau_{j+2})^{n-1}))\leq\frac{20}{n-1}\mu(r_{pq}(\tau_1,\dots,\tau_j,1,\tau_{j+2})^{n-1}).$$

By construction

$$\mu(r_{pq}(\tau_1,\dots,\tau_j,1,\tau_{j+2})^{n-1})=\mu(r_{pq}(\tau_1,\dots,\tau_j,1,\tau_{j+2}))=\mu(r_{pq}(\tau_1,\dots,\tau_j,1,0))=l,$$

so

$$\lambda(f(E_{pq}))\leq 2\frac{20}{n-1}l.$$

Using this and the definition of K the measure of that part of $f(M(\tau_1,\dots,\tau_j,1)^n)$ that is uncovered by $f(M(\tau_1,\dots,\tau_j,0)^n)$ is at most $K\lambda(f(E_{pq}))\leq K\frac{40}{n-1}l$. Therefore

$$\lambda(f(M(\tau_1,\dots,\tau_j)^{n+1}))\leq\lambda(f(M(\tau_1,\dots,\tau_j,0)^n)+K\frac{40}{n-1}l. \tag{12}$$

In the remaining part of the proof we only need to make calculations showing that with our conditions the estimation (12) implies the statement. That is we need to show that the right-hand side of (12) is at most $\frac{20}{n+1}L$.

Using the induction assumption and (5)

$$\lambda(f(M(\tau_1,\dots,\tau_j,0)^n)\leq\frac{20}{n}\mu(M(\tau_1,\dots,\tau_j,0)^n)=\frac{10}{n}L. \tag{13}$$

Let us find an upper bound for Kl. For this we need some simple estimations:

By construction

$$\frac{d_0}{2^j} = \mu(M(\tau_1, \dots, \tau_j)) = \mu(M(\tau_1, \dots, \tau_j)^{n+1}) = L. \tag{14}$$

Using this, the formula (2) of $\delta(\tau_1, \dots, \tau_j)$ and $n \leq N$ we obtain

$$\delta k_0 k_1(\tau_1) \dots k_j(\tau_j) = \frac{d_0}{m_j 2^{j+2}} = \frac{L}{4N} \leq \frac{L}{4n}. \tag{15}$$

Similarly, using the formula (1) for $l(\tau_1, \dots, \tau_j, 1, 0)$ we obtain

$$l k_0 k_1(\tau_1) \dots k_j(\tau_j) = \frac{d_0}{2^{j+2} k_{j+1}(1)} = \frac{L}{4k_{j+1}(1).} \tag{16}$$

By the condition $(**)$, using $j + n + 1 \leq N$ we get

$$J_n(\tau_1, \dots, \tau_j, 0) \leq J_{N-j-1}(\tau_1, \dots, \tau_j, 0) \leq \frac{k_{j+1}(1)}{N} \leq \frac{k_{j+1}(1)}{n}. \tag{17}$$

Replacing T by the estimation (10) in the formula (11) and multiplying by $4l$, then using the estimations (15-17) and the condition of the Case II we obtain

$$\begin{aligned} 4Kl &\leq 4lk_0 k_1(\tau_1) \dots k_j(\tau_j) k_{j+1}(1) - 2\lambda(f(M(\tau_1, \dots, \tau_j, 0)^n)) + \\ &+ 16\delta k_0 k_1(\tau_1) \dots k_j(\tau_j) + 8lk_0 k_1(\tau_1) \dots k_j(\tau_j) J_n(\tau_1, \dots, \tau_j, 0) \leq \\ &\leq L - 2\frac{9}{n}L + \frac{16L}{4n} + \frac{8L}{4k_{j+1}(1)} \frac{k_{j+1}(1)}{n} = \\ &= L\left(1 - \frac{18}{n} + \frac{4}{n} + \frac{2}{n}\right) = \\ &= L\left(1 - \frac{12}{n}\right). \end{aligned} \tag{18}$$

Finally using (13) and (18) in (12), with trivial estimations and transformations we obtain the inequality we wanted to prove:

$$\begin{aligned} \lambda(f(M(\tau_1, \dots, \tau_j)^{n+1})) &\leq \frac{10}{n}L + L\left(1 - \frac{12}{n}\right)\frac{10}{n-1} = 10L\left(\frac{1}{n} + \frac{1}{n-1} - \frac{12}{n(n-1)}\right) \leq \\ &\leq 10L\left(\frac{1}{n} + \frac{1}{n-1} - \frac{3}{n(n-1)}\right) = 10L\frac{2n-4}{n(n-1)} = 20L\frac{1}{n}\frac{n-2}{n-1} \leq \\ &\leq 20L\frac{1}{n}\frac{n}{n+1} = \frac{20}{n+1}L. \quad \square \end{aligned}$$

4. Now we are ready to construct a proper set M_0^*. Let $d_0 = 1$, that is the initial segment M has length 1. We will denote this segment by M_0. We are constructing the parameters $k_j(\tau)$ and m_j in the following way:

Let $N_1, N_2, \ldots$ be an increasing sequence of integers greater than 1. (We will choose this sequence later.) For all N_s take the parameters constructed in Proposition 3.5, that is let

$$m_j^s = N_s \ (j = 0, 1, \ldots, N_s - 1), \quad k_0^s = N_s, \quad k_j^s(0) = N_s \ (j = 1, 2, \ldots, N_s - 1),$$

and let $k_j^s(1)$ $(j = 1, 2, \ldots, N_s - 1)$ be chosen such a way that $(*)$ and $(**)$ holds for the parameters with upper index s. Then let the sequences $k_j(\tau)$ and m_j be these sequences successively, that is let

$$\begin{aligned} m_{N_1+N_2+\ldots+N_{s-1}+i} &= m_i^s (= N_s) \qquad (i = 0, 1, \ldots, N_s - 1;\ s \geq 1) \\ k_{N_1+N_2+\ldots+N_{s-1}}(\tau) &= k_0^s (= N_s) \qquad (s \geq 1;\ \tau = 0, 1) \\ k_{N_1+N_2+\ldots+N_{s-1}+i}(\tau) &= k_i^s(\tau) \qquad (i = 1, 2, \ldots, N_s - 1;\ s \geq 1;\ \tau = 0, 1). \end{aligned}$$

Note that using these parameters in the construction means the following: First we apply the transformation $M \to M^N$ of Proposition 3.6 to the unit segment M_0 with $N = N_1$. Then we apply this transformation to the horizontal segments we got with $N = N_2$, then to the new segments with $N = N_3$,...etc

Proposition 4.1

$$\mu(M_0^*) \geq \frac{1}{20}, \quad \text{and what is more} \quad c(M_0^*) \geq \frac{1}{20},$$

if M_0^* is constructed by the above described way.

Proof. According to Proposition 2.4 it is enough to check validity of the condition $(*)$. That holds for the indexes $j = N_1 + N_2 + \ldots + N_s$, since

$$k_{N_1+N_2+\ldots+N_s}(\tau) = N_{s+1} > N_s = m_{N_1+N_2+\ldots+N_{s-1}+(N_s-1)}.$$

Condition $(*)$ also holds for the other indexes, since we made them so. □

Now we only have to prove that for a proper (rapidly increasing) sequence N_s the set M_0^* cannot be mapped onto a segment by contraction. For this we are parallel constructing the sequences of parameters N_s and σ_s such that

$$(***) \qquad \begin{gathered} \text{for arbitrary positive integer } s \text{ and contraction } f : \mathbf{R}^2 \to \mathbf{R} \\ U_{\sigma_s}(M_0^{N_1+\ldots+N_s}) \supset M_0^{N_1+\ldots+N_r} \ (\text{if } r > s) \quad \text{and} \quad \lambda(U_{\sigma_s}(f(M_0^{N_1+\ldots+N_s}))) \leq \frac{21}{s} \end{gathered}$$

First we prove that this is enough.

Proposition 4.2 *If the parameters $1 < N_1 < N_2 < \ldots$, $\sigma_1, \sigma_2, \ldots$ satisfy the condition $(***)$ then M_0^* cannot be mapped onto a segment by contraction.*

Proof. Suppose that $f_0 : M_0^* \to \mathbf{R}$ is such a contraction. Then $\lambda(f_0(M_0^*)) > 0$. Using Kirszbraun theorem (see e.g. in [2]) f_0 can be extended to an $f : \mathbf{R}^2 \to \mathbf{R}$ contraction. Obviously $\lambda(f(M_0^*)) = \lambda(f_0(M_0^*)) > 0$. Let s be so big that

$$\lambda(f(M_0^*)) > \frac{21}{s}.$$

We are proving from $(***)$ that

$$M_0^* \subset U_{\sigma_s}(M_0^{N_1+\ldots+N_s}). \tag{19}$$

Suppose that $x \in M_0^* \setminus U_{\sigma_s}(M_0^{N_1+\ldots+N_s})$. Since $U_{\sigma_s}(M_0^{N_1+\ldots+N_s})$ is closed there exists an $\varepsilon > 0$ such that $x \notin U_\varepsilon(U_{\sigma_s}(M_0^{N_1+\ldots+N_s}))$. Using $(***)$ this implies that $x \notin U_\varepsilon(M_0^{N_1+\ldots+N_r})$ if $r > s$. But since $M_0^{N_1+\ldots+N_r}$ tends to M_0^* $(r \to \infty)$ by the Hausdorff metric, this implies that $x \notin M_0^*$ which is a contradiction.

Since f is contraction, (19) implies that

$$f(M_0^*) \subset f(U_{\sigma_s}(M_0^{N_1+\ldots+N_s})) \subset U_{\sigma_s}(f(M_0^{N_1+\ldots+N_s})).$$

This is a contradiction since $\lambda(f(M_0^*)) > \frac{21}{s}$ but according to the condition $(***)$,

$$\lambda(U_{\sigma_s}(f(M_0^{N_1+\ldots+N_s}))) \leq \frac{21}{s}. \ \square$$

Therefore now we only have to choose sequences N_s and σ_s with the property $(***)$. For this we are proving two statements:

Proposition 4.3

$$\lambda(U_\sigma(f(M_0^{N_1+\ldots+N_s}))) \leq \frac{20}{N_s} + 2\sigma J^s$$

for arbitrary $\sigma > 0$, positive integer s and $f : \mathbf{R}^2 \to \mathbf{R}$ contraction, where J^s denotes the number of the segments of $M_0^{N_1+\ldots+N_s}$.

Proof. The set $M_0^{N_1+\ldots+N_{s-1}}$ consists of horizontal segments and the sum of their length is 1. We constructed $M_0^{N_1+\ldots+N_s}$ applying transformation $M \to M^N$ of Proposition 3.6 with its conditions and $N = N_s$ to each segment of $M_0^{N_1+\ldots+N_{s-1}}$. Therefore using Proposition 3.6 we obtain

$$\lambda(f(M_0^{N_1+\ldots+N_s})) \leq \frac{20}{N_s}.$$

Since $f(M_0^{N_1+\ldots+N_s})$ consists of J^s segments (or points) the measure of its neighborhood with radius σ is at most by $2\sigma J^s$ greater than the measure of itself. This concludes the proof. $\square$

Proposition 4.4

$$U_{\frac{1}{2N_{s+1}}}(M_0^{N_1+\ldots+N_s}) \supset M_0^{N_1+\ldots+N_r}, \quad \text{if } r > s.$$

Proof. The set $M_0^{N_1+\ldots+N_r}$ is obtained by applying the transformation $P_{k,m}$ to the segments of $M_0^{N_1+\ldots+N_s}$ $N_{s+1}+\ldots+N_r$ times. We are going to prove that the part of $M_0^{N_1+\ldots+N_r}$ originated from a fixed segment of $M_0^{N_1+\ldots+N_s}$ is in the (closed) neighborhood of the segment with radius $\frac{1}{2N_{s+1}}$.

The part originated from a fixed segment is above the segment. So it is enough to prove that a point of a segment of $M_0^{N_1+\ldots+N_s}$ cannot go upward more than $\frac{1}{2N_{s+1}}$ during the process.

By the definition of $P_{k,m}$ during a transformation $P_{k,m}$ a point of a segment can go upward at most the $4km$-th part of the length of the segment. So since $4k_0^{s+1}m_0^{s+1} \geq 4m_0^{s+1} = 4N_{s+1}$, at the first step a point of a segment of $M_0^{N_1+\ldots+N_s}$ can go upward at most the $4N_{s+1}$-th part of the length of the segment. Since the segments are divided into at least 2 parts in each step a point can go upward at most the $2N_{s+1}$-th part of the length of the original segment after arbitrary many steps. On the other hand the length of any segment of $M_0^{N_1+\ldots+N_s}$ is at most 1 (indeed much smaller). Therefore a point of $M_0^{N_1+\ldots+N_s}$ can go upward at most $\frac{1}{2N_{s+1}}$, which concludes the proof. □

The previous 3 statements show that now we only have to choose sequences N_s and σ_s with the following conditions:

(i)
$$1 < N_1 < N_2 < N_3 < \ldots$$

(ii)
$$\frac{1}{2N_{s+1}} \leq \sigma_s$$

(iii)
$$\frac{20}{N_s} + 2\sigma_s J^s \leq \frac{21}{s}$$

$$(s = 1, 2, \ldots).$$

If (i) holds, then $N_s \geq s$, so $\frac{20}{N_s} \leq \frac{20}{s}$. Therefore (iii) can be replaced by the condition $2\sigma_s J^s \leq \frac{1}{s}$. We can construct such a sequence by recursion:

Let N_1 be an integer greater than 1. If $N_1, \ldots, N_s$; $\sigma_1, \ldots, \sigma_{s-1}$ are already defined $(s \geq 1)$, then let

$$\sigma_s = \frac{1}{2sJ^s}.$$

Then let N_{s+1} be an integer greater than $\frac{1}{2\sigma_s}$ and N_s. (We used that J^s depends only on $N_1, \ldots, N_s$.)

The constructed numbers obviously satisfy the desired conditions. So according to the previous 3 statements if we use these parameters then M_0^* cannot be mapped onto a segment by contraction. On the other hand according to Theorem 4.1 its linear measure is at least $\frac{1}{20}$.

Therefore M_0^* is indeed a compact set in the plane with positive linear measure but it cannot be mapped onto a segment by contraction, so we proved the Main Theorem.

Corollary 1. *The Kolmogoroff minimal linear measure and the linear Hausdorff measure are incommensurable.*

Corollary 2. *If $n \geq 2$ and $k = 1$ then it is not true that every (k-dimensional Hausdorff) measurable subset of $\mathbf{R}^n$ with positive k-dimensional Hausdorff measure can be mapped, using a contraction onto a segment.*

Remark. The author does not know anything in the missing cases, that is when $n \geq 3$ and $2 \geq k \geq n$. (The case n=k=1 and n=k=2 is mentioned in the introduction.)

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