Construction of 1-dimensional subsets of the reals not containing similar copies of given patterns

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Abstract

For any countable collection of sets of three points we construct a compact subset of the real line with Hausdorff dimension 1 that contains no similar copy of any of the given triplets.

1 Introduction

An old conjecture of Erdős [6] (which is also known as Erdős similarity problem) states that for any infinite set $A \subset \mathbb{R}$ there exists a set $E \subset \mathbb{R}$ of positive Lebesgue measure which does not contain any similar (i.e. translated and rescaled) copy of A. It is known that slowly decaying sequences are not counterexamples [8, 2, 12] (see e.g. [10, 13, 16] for other related results) but nothing is known about any infinite sequence that converges to zero at least exponentially. On the other hand, it follows easily from Lebesgue's density theorem that any set $E \subset \mathbb{R}$ of positive Lebesgue measure contains similar copies of every finite sets.

Bisbas and Kolountzakis [1] gave a noncomplete proof of the following related statement: For every infinite set $A \subset \mathbb{R}$ there exists a compact set $E \subset \mathbb{R}$ of Hausdorff dimension 1 such that E contains no similar copy of A. Kolountzakis asked whether the same holds for finite sets as well. Iosevich asked a similar question: if $A \subset \mathbb{R}$ is a finite set and $E \subset [0,1]$ is a set of given Hausdorff dimension, must E contain a similar copy of A?

In this paper we answer these questions by showing that for any set $A \subset \mathbb{R}$ of at least 3 elements there exists a 1-dimensional set that contains no similar copy of A. In fact, we prove a bit more by proving the following theorem, which immediately yields the following two corollaries.

Theorem 1. *For any countable set $A \subset (1,\infty)$ there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 such that if $x < y < z, x, y, z \in E$ then $\frac{z-x}{z-y} \notin A$.*

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Corollary 2. *For any sequence $B_1, B_2, \ldots \subset \mathbb{R}$ of sets of at least three elements there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 that contains no similar copy of any of $B_1, B_2, \ldots$.*

Corollary 3. *For any countable set $B \subset \mathbb{R}$ there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 that intersects any similar copy of B in at most two points.*

The method of the construction is similar to the method used in [11], where a compact set A of Hausdorff dimension 1 is constructed such that A does not contain any set of the form $\{a, a+b, a+c, a+b+c\}$ for any $a, b, c \in \mathbb{R}; b, c \neq 0$, so in particular A does not contain any non-trivial 3-term arithmetic progression.

Laba and Pramanik [14] obtained a positive result by proving that if a compact set $E \subset \mathbb{R}$ has Hausdorff dimension sufficiently close to 1 and E supports a probability measure whose Fourier transform has appropriate decay at infinity then E must contain non-trivial 3-term arithmetic progressions. It would be interesting to know whether similar conditions could guarantee other finite patterns as well.

Perhaps one can even find conditions weaker than having positive measure that implies that a compact subset of $\mathbb{R}$ contains similar copies of all finite subsets. This is not impossible since Erdős and Kakutani [7] constructed a compact set of measure zero with this property. The Erdős-Kakutani set has Hausdorff dimension 1 but, using ideas from [5], Máthé [15] constructed such a set with Hausdorff dimension 0. However, the packing dimension of such a set must be 1, since the argument of the proof of [3, Theorem 2] gives that if a compact set $C \subset \mathbb{R}$ contains similar copies of all sets of n points then C has packing dimension at least $(n-2)/n$.

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2 Proof of Theorem 1

Fix a sequence $\alpha_1, \alpha_2, \ldots \subset A$ so that each element of A appears infinitely many times in the sequence (α_k). Let

$$\beta_k = \max\left(6\alpha_k, \frac{6\alpha_k}{\alpha_k - 1}\right) \qquad (k \in \mathbb{N}). \tag{1}$$

Since $A \subset (1, \infty)$, the number β_k is defined and $\beta_k > 6$ for every k. We can clearly choose a sequence $m_1, m_2, \ldots \subset \{3, 4, 5, \ldots\}$ so that

$$\lim_{k\to\infty} \frac{\log(\beta_1 \cdot \ldots \cdot \beta_k)}{\log(m_1 \cdot \ldots \cdot m_{k-1})} = 0. \tag{2}$$

Let

$$\delta_k = \frac{1}{\beta_1 \cdot \ldots \cdot \beta_k \cdot m_1 \cdot \ldots \cdot m_k}. \tag{3}$$

By induction we shall define sets $E_0 \supset E_1 \supset E_2 \supset \dots$ such that for each $k \in \mathbb{N}$

(*) E_k consists of $m_1 \cdot \ldots \cdot m_k$ closed intervals of length δ_k which are separated by gaps of at least δ_k and each interval of E_{k-1} contains m_k intervals of E_k.

We will denote by $I_1^k, I_2^k, \dots, I_{m_1 \cdots m_k}^k$ the intervals of E_k ordered from left to right, and by $(J_n, K_n, L_n)_{n \in \mathbb{Z}}$ an enumeration of the set

$$\Gamma = \{(I_a^k, I_b^k, I_c^k) : a, b, c, k \in \mathbb{N}, a < b < c \le m_1 \cdot \ldots \cdot m_k\}$$

such that if $n > 1$ and $(J_n, K_n, L_n) = (I_a^k, I_b^k, I_c^k)$ then $n > k$. Since each element of A appears infinitely many times in the sequence (α_k), by repeating each element of Γ infinitely many times we can also guarantee that

$$(\forall a \in A)\ (\forall (J, K, L) \in \Gamma)\ (\exists n \in \mathbb{N})\ \alpha_n = a,\ (J_n, K_n, L_n) = (J, K, L). \qquad (4)$$

Let $E_0 = [0, 1]$ and choose E_1 so that (*) holds for $k = 1$. Suppose that $k \ge 2$ and $E_1, \dots, E_{k-1}$ is already defined so that (*) holds for $1, \dots, k-1$. Then (J_k, K_k, L_k) is already defined and each interval of E_{k-1} is either contained in exactly one of J_k, K_k and L_k or disjoint from them.

We shall define E_k so that $x \in E_k \cap J_k$, $y \in E_k \cap K_k$ and $z \in E_k \cap L_k$ will imply that $\frac{z-x}{z-y} \ne \alpha_k$.

Let I be an interval of E_{k-1} which is contained in J_k. Since I has length δ_{k-1} and using (3) and (1) we have

$$\frac{\delta_{k-1}}{3\alpha_k \delta_k} = \frac{m_k \beta_k}{3\alpha_k} \ge 2m_k > m_k + 1,$$

I contains more than m_k points of the form $3\alpha_k \delta_k i$ ($i \in \mathbb{Z}$). Hence we can choose the m_k intervals of E_k in I as segments of the form $\delta_k(3i\alpha_k + [0, 1])$ ($i \in \mathbb{Z}$).

If I is an interval of E_{k-1} which is contained in K_k then similarly, since $\frac{\delta_{k-1}}{3\delta_k} = \frac{m_k \beta_k}{3} \ge 2m_k > m_k + 1$ we can choose the m_k intervals of E_k in I as segments of the form $\delta_k(3j + [0, 1])$ ($j \in \mathbb{Z}$).

If I is an interval of E_{k-1} which is contained in L_k then, since by (3) and (1) we have

$$\frac{\delta_{k-1}}{\frac{3\alpha_k}{\alpha_k - 1}\delta_k} = \frac{m_k \beta_k}{\frac{3\alpha_k}{\alpha_k - 1}} \ge 2m_k > m_k + 1,$$

we can choose the m_k intervals of E_k in I as segments of the form $\delta_k(\frac{3\alpha_k}{\alpha_k - 1}(l + \frac{1}{2}) + [0, 1])$ ($l \in \mathbb{Z}$).

In each of the rest of the intervals of E_{k-1} we define the m_k intervals of length δ_k of E_k arbitrarily so that they are separated by gaps of at least length δ_k.

This way we defined E_k so that (*) holds. Let $E = \cap_{k=1}^{\infty} E_k$. Then E is clearly a compact subset of $\mathbb{R}$. Condition (*) implies that the Hausdorff dimension of E is at least

$$\liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)},$$

cf. [9] Example 4.6. On the other hand, using (3) and (2) we get that

$$\liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_k)} = \liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{\log(\beta_1 \cdots \beta_k) + \log(m_1 \cdots m_{k-1})} = 1,$$

therefore the Hausdorff dimension of E is 1.

Finally, to get a contradiction, suppose that $x, y, z \in E$, $x < y < z$ and $(z-x)/(z-y) \in A$. Since $\delta_k \to 0$, there exists a $k \in \mathbb{N}$ such that x, y and z are in distinct intervals of E_k. Then, by (4) there exists an $n \in \mathbb{N}$ so that $x \in J_n, y \in K_n, z \in L_n$ and $(z-x)/(z-y) = \alpha_n$. By the construction of E_n, there exists $i, j, l \in \mathbb{Z}$ such that

$$x \in \delta_n(3i\alpha_n + [0,1]),\ y \in \delta_n(3j + [0,1]),\ z \in \delta_n\left(\frac{3\alpha_n}{\alpha_n - 1}(l + 1/2) + [0,1]\right).$$

Let

$$X = 3i\alpha_n + [0,1], \qquad Y = 3j + [0,1], \quad Z = \frac{3\alpha_n}{\alpha_n - 1}(l+1/2) + [0,1].$$

Then $x/\delta_n \in X$, $y/\delta_n \in Y$ and $z/\delta_n \in Z$. On the other hand, $\frac{z-x}{z-y} = \alpha_n$ implies that $\alpha_n y = x + (\alpha_n - 1)z$, so (by using the notation $A + B = \{a + b : a \in A, b \in B\}$) we must have

$$\alpha_n Y \cap (X + (\alpha_n - 1)Z) \neq \emptyset. \tag{5}$$

By definition (and using that $\alpha_n > 1$),

$$\alpha_n Y = \alpha_n(3j + [0,1]) \qquad \text{and} \tag{6}$$

$$\begin{aligned} X + (\alpha_n - 1)Z &= 3i\alpha_n + [0,1] + 3\alpha_n(l + 1/2) + (\alpha_n - 1)[0,1] \\ &= 3(i+l)\alpha_n + [3\alpha_n/2, 5\alpha_n/2] \\ &= \alpha_n(3(i+l) + [3/2, 5/2]). \end{aligned} \tag{7}$$

Since $i, j, l \in \mathbb{Z}$, (6) and (7) contradict (5).

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