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COVERING $\mathbb{R}$ WITH TRANSLATES OF A COMPACT SET

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ABSTRACT. Motivated by a question of Gruenhage, we investigate when $\mathbb{R}$ is the union of less than continuum many translates of a compact set $C \subseteq \mathbb{R}$. It will follow from one of our results that if a compact set C has packing dimension less than 1, then $\mathbb{R}$ is not the union of less than continuum many translates of C.

1. INTRODUCTION

When is $\mathbb{R}$ the union of less than continuum many translates of a given compact subset of $\mathbb{R}$? Of course, if the compact set has non-empty interior, then $\mathbb{R}$ is easily seen to be the union of countably many translates of the compact set. On the other hand, if we assume the continuum hypothesis, then it follows from the Baire category theorem that there is no such nowhere dense compact set.

Gary Gruenhage observed that it is consistent with ZFC that given a compact set of positive Lebesgue measure one can find less than continuum many translates of it whose union is $\mathbb{R}$. Hence, for nowhere dense compact sets of positive Lebesgue measure the question whether $\mathbb{R}$ can be written as less than continuum many translates of the given set is independent of ZFC.

Gruenhage also showed that $\mathbb{R}$ is not the union of less than continuum many translates of the standard "middle 1/3 Cantor set". Motivated by these results, he asked the following natural question:

Problem 1.1. *Is there a compact set of Lebesgue measure zero and less than continuum many translates of it whose union is $\mathbb{R}$?*

Of course, a positive answer to this problem would require some extra set-theoretic assumption.

For the sake of notational convenience, let us call a compact set $C \subseteq \mathbb{R}$ *thin* if it is true in ZFC that $\mathbb{R}$ is not the union of less than continuum many translates of C. Hence, Gruenhage's question is whether every compact set of Lebesgue measure zero is thin.

Daniel Mauldin asked whether at least the compact sets with Hausdorff dimension less than 1 are thin. In this note, we show that if we consider packing dimension instead of Hausdorff dimension then the answer is affirmative.

Ronnie Levy asked whether it is true that $\mathbb{R}$ is not the union of less than continuum many similar copies of the standard middle 1/3 Cantor set. We show that the answer is affirmative. We call a set *similarity thin* if it satisfies the definition of thin with word "translates" replaced by "similar copies". We show that compact

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sets with packing dimension less than 1 are similarity thin. A more general result will be obtained, too. Finally, we shall see that Problem 1.1 would be independent of ZFC if we wanted a G_δ-set instead of a compact set.

2. Results

For the sake of completeness, we first prove the following theorem due to Gruenhage. Proof given here is due to Márton Elekes.

Theorem 2.1. *(Gruenhage) It is consistent with ZFC that given a compact set of positive measure, one can find less than continuum many translates of it whose union is $\mathbb{R}$.*

Proof. (M. Elekes) It is consistent with ZFC that there is a set $A \subseteq \mathbb{R}$ of cardinality less than the continuum which has positive Lebesgue outer measure. (See e.g. in [1].) Let C be a compact set of positive measure. By a variant of the well known theorem of Steinhaus the sum of a measurable set with positive measure and a set with positive outer measure contains an interval. Hence, $A + C$ contains an interval. Now, let $T = \mathbb{Q} + \mathbb{A}$. Then, T is a set with cardinality less than that of the continuum and $T + C = \mathbb{R}$. □

The basic idea behind our main result is the following simple fact. Recall that set $A \subseteq \mathbb{R}$ is similar to set B if there are numbers s, t, $s \neq 0$ such that $B = t + s \cdot A$.

Lemma 2.2. *Let C be a compact set. If there is a perfect set P such that $(t + s \cdot C) \cap P$ is countable for every t and every $s \neq 0$, then C is similarity thin.*

If $C \subseteq \mathbb{R}$, then $C^n = \{(p_1, p_2, \dots, p_n) : p_i \in C \text{ for all } 1 \le i \le n\}$. If $A \subseteq \mathbb{R}^n$, then $A_* = \{(x_1, x_2, \dots, x_n) \in A : (i \neq j) \implies (x_i \neq x_j)\}$. We define $F_n : \mathbb{R}^{n+2} \to \mathbb{R}^n$ by $F_n(x_1, x_2, \dots, x_n, s, t) = (t + sx_1, \dots, t + sx_n)$.

Lemma 2.3. *Suppose that C and P are compact sets. If a similar copy of C intersects P in at least n points, then $F_n(C^n \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cap P^n_* \neq \emptyset$.*

Proof. Let $y_1, y_2, \dots, y_n$ be n distinct points of P which are contained in some similar copy of C. Then $(y_1, \dots, y_n) \in F(C^n \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cap P^n_*$. □

Lemma 2.4. *Fix a positive integer n. Suppose that $M \subseteq \mathbb{R}^n$ is an F_σ, first category set. Then, there is a perfect set $P \subseteq \mathbb{R}$ such that $P^n_* \cap M = \emptyset$.*

Proof. Let $M = \cup_{i=1}^\infty M_i$ where M_i's are compact nowhere dense sets and $M_i \subseteq M_{i+1}$. Set P will be the intersection of certain open sets which we define inductively. At stage $k \ge 0$, we will have 2^k many intervals. At stage $k = 0$, let $U_\emptyset$ be an open interval of $\mathbb{R}$ with diameter less than 1. Suppose that we are at stage k, U_σ has been defined for $\sigma \in \{0,1\}^k$ and the following properties hold.

1. For each $\sigma \in \{0,1\}^i$, $1 \le i \le k$, U_σ is an open interval of $\mathbb{R}$ with length less than 2^{-i}.
2. If $\sigma \in \{0,1\}^i$ for some $1 \le i \le k-1$ and $t \in \{0,1\}$, then $\overline{U_{\sigma t}} \subseteq U_\sigma$.
3. If $\sigma, \tau \in \{0,1\}^i$ with $\sigma \neq \tau$, then $U_\sigma \cap U_\tau = \emptyset$.
4. If $1 \le i \le k$ and $x_j \in \sigma_j \in \{0,1\}^i$ with σ_j's all distinct for $1 \le j \le n$, then $(x_1, x_2, \dots, x_n) \notin M_i$.

We now define intervals U's at stage $k+1$. Fix $\sigma \in \{0,1\}^k$. Let $V^0_{\sigma 0}, V^0_{\sigma 1}$ be pairwise disjoint open intervals with length less than $2^{-(k+1)}$ whose closure is a subset of U_σ. Let $\mathcal{V}^0 = \{V^0_{\sigma t} : \sigma \in \{0,1\}^k \text{ and } t \in \{0,1\}\}$. Our U's at stage $k+1$ will be a refinement of collection $\mathcal{V}^0$. It is clear that any refinement will satisfy properties (1)-(3). We list all 1-1 functions from $\{1,2,\dots,n\}$ into $\{0,1\}^{k+1}$ as g_l, $1 \leq l \leq L$, where $L = \frac{2^{k+1}!}{(2^{k+1}-n)!}$. We induct on l. At $l = 1$, let $\mathcal{V}^1 = \{V^1_\sigma\}_{\sigma\in\{0,1\}^{k+1}}$ be such that

- for each $\sigma \in \{0,1\}^{k+1}$, V^1_σ is a nonempty open subinterval of V^0_σ, and
- if $x_i \in V^1_{g_1(i)}$ for each $1 \leq i \leq n$, then $(x_1, x_2, \dots, x_n) \notin M_{k+1}$.

This is possible since $V^0_{g_1(1)} \times \dots \times V^0_{g_1(n)}$ is an open subset of $\mathbb{R}^n$ and M_{k+1} is a nowhere dense closed subset of $\mathbb{R}^n$. Similarly, we may obtain a collection $\mathcal{V}^2 = \{V^2_\sigma\}_{\sigma\in\{0,1\}^{k+1}}$ which is a refinement of $\mathcal{V}^1$ so that if $x_i \in V^2_{g_2(i)}$ for $1 \leq i \leq n$, then $(x_1, x_2, \dots, x_n) \notin M_{k+1}$. We continue construction of $\mathcal{V}^l$ in this fashion until we reach $\mathcal{V}^L$. Collection $\mathcal{V}^L$ is the desired collection, ie $\{U_\sigma\}_{\sigma\in\{0,1\}^{k+1}}$ satisfies conditions (1)-(4), where $U_\sigma = V^L_\sigma$. This completes the induction.

Now, it is easy to verify that $P = \cap_{k=1}^{\infty}(\cup_{\sigma\in\{0,1\}^k} U_\sigma)$ is the desired perfect set. □

Let us recall some terminology and facts from basic geometric measure theory. If $A \subseteq \mathbb{R}^n$, then $dim_p(A)$ and $dim_h(A)$ denote the packing dimension and the Hausdorff dimension of A, respectively. (See the definitions and basic properties e.g. in [2])

Theorem 2.5. *Every compact subset of $\mathbb{R}$ with packing dimension less than 1 is similarity thin. That is, less than continuum many similar copies of a compact set with packing dimension less than 1 cannot cover the real line.*

Proof. Let $C \subseteq \mathbb{R}$ be a compact set with packing dimension less than 1. By Lemma 2.2, it will suffice to show that there is a perfect set P such that $(t+s\cdot C)\cap P$ is finite for all real t, s with $s \neq 0$.

Recall (see e. g. in [2]) that for the packing dimension, we have that for Borel sets A, B, $dim_h(A \times B) \leq dim_p(A \times B) \leq dim_p(A) + dim_p(B)$. Hence, we may choose n sufficiently large so that $dim_p(C^n) < n - 2$, which, in turn, implies that $dim_p(C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}) < n$ and hence $dim_h(C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}) < n$. We note that F_n is countably Lipschitz, ie we can decompose the domain of F_n into countably many compact set $\{A_i\}$ so that $F_n|A_i$ is Lipschitz. Let $B_i = (C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}) \cap A_i$. Then, since Lipschitz maps clearly cannot increase the Hausdorff dimension, we have that $F_n(B_i)$ is a compact set with $(n-2)$ dimensional Lebesgue measure zero and hence of first category. However, $F_n(C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}) = \cup_{i=1}^{\infty} F_n(B_i)$. Therefore, $F_n(C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R})$ is an F_σ first category set. By Lemma 2.4, we have that there is a perfect set $P \subseteq \mathbb{R}$ such that $P^n_* \cap F_n(C^n \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}) = \emptyset$. By Lemma 2.3, we have that no similar copy of C intersects P in more than n points. By Lemma 2.2, we have that C is similarity thin. □

Theorem 2.5 can be easily generalized to any countably Lipschitz, finite (say, k) parameter images instead of similar copies. One can easily check that by replacing $(x, s, t) \to t + sx$ by any other countably Lipschitz function $f : \mathbb{R} \times H \to \mathbb{R}$, where $H \subseteq \mathbb{R}^k$ replaces $(\mathbb{R}\setminus\{0\}) \times \mathbb{R}$ as the set of possible parameters and replacing 2 by k at some points, the same way as above, we can get the following generalization.

Theorem 2.6. *Let $H \subseteq \mathbb{R}^k$ be a set of possible parameters, let $f : \mathbb{R} \times H \to \mathbb{R}$ be a countably Lipschitz function. If C is a compact set with packing dimension less* *than* 1 *then less than continuum many sets of the form* $f(C,h)$ $(h \in H)$ *cannot cover the real line.*

3. REMARKS

As in Lemma 2.2, a negative answer to the original Problem 1.1 would follow from a negative answer to the following question:

Problem 3.1. *Is there a compact set* C *of Lebesgue measure zero such that every perfect set intersects at least one of the translates of* C *in uncountably many points?*

A positive answer to this question would only show that this method cannot solve Problem 1.1. But to this problem one can imagine (contrasted to Problem 1.1) a positive answer in ZFC.

Both problems seem just as hard if "compact" is replaced by "F_σ". However, it is consistent with ZFC that there is a set of Lebesgue measure zero and less than continuum many translates of it whose union is $\mathbb{R}$ (see e.g. in [1]). In fact, if there exists a set of second category with cardinality less than the continuum (which is consistent with ZFC, see e.g. in [1]) then any residual set of Lebesgue measure zero has this property. (Since, as one can easily check, the sum of a set of second category and a residual set is $\mathbb{R}$.) Since there exist residual G_δ sets with Lebesgue measure zero, this means that if we replaced "compact" in Problems 1.1 and 3.1 by "G_δ" then the positive answers would be consistent with ZFC.

REFERENCES

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