

Tiling of rectangles with squares and related problems via Diophantine approximation



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ABSTRACT

This article shines new light on the classical problem of tiling rectangles with squares efficiently with a novel method. With a twist on the traditional approach of resistor networks, we provide new and improved results on the matter using the theory of Diophantine Approximation, hence overcoming long-established difficulties, such as generalizations to higher-dimensional analogues. The universality of the method is demonstrated through its applications to different tiling problems. These include tiling rectangles with other rectangles, with their respective higher-dimensional counterparts, as well as tiling equilateral triangles, parallelograms, and trapezoids with equilateral triangles.

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1. Introduction

Tiling rectangles by squares has been a popular topic for more than 100 years. The first fundamental result is the following.

Theorem 1.1 (M. Dehn [4], 1903). *If a rectangle is tiled by squares then the aspect ratio of the rectangle is rational.*

A variety of proofs of Theorem 1.1 have been discovered since then, with a particular standout being from a paper by R.L. Brooks et al. [1] in 1940. This paper presents a one-to-one correspondence between a squared rectangle and a resistor network, with any solution to Kirchoff's equations being exactly the sizes of the square tiles. Using the physical theory of

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² In memory of Xianchun Meng.

resistor networks, the problem is easily transformed into a graph theoretic one. In fact, Dehn proved a stronger version of Theorem 1.1 (later referred to as “Strong Dehn’s Theorem”) [4]: If a rectangle is tiled by squares then the side lengths of the squares are all rational multiples of the side lengths of the rectangle.

A natural next step of development is to investigate the possible rational numbers that could represent the ratio of the sides of a particular tiling; for example, is there anything one could conclude about the ratio given the number of tiles used? Specifically, two versions of this problem are well-studied historically:

- (1) [Coprime Rectangle Tiling] What is the minimal number of square tiles of integer side lengths needed to tile a rectangle of size $p \times q$, where $p, q \in \mathbb{Z}$ and $(p, q) = 1$?
- (2) [Prime Tiling, also known as *Mrs Perkins’s quilt*] What is the minimal number of square tiles of jointly coprime (integer) side lengths needed to tile a rectangle of size $p \times q$, where $p, q \in \mathbb{Z}$?

It should be more or less clear that Prime Tiling is a more general problem than Coprime Rectangle Tiling, as any integer tiling of a coprime rectangle $p \times q$ cannot have common factors among the side lengths of its tiles: otherwise this would also be a common factor between p and q . Therefore, any result on Prime Tiling automatically applies to Coprime Rectangle Tiling. For each of these two questions, many lower and upper bounds on the number of tiles needed were established in the past endeavors.

Regarding the Coprime Rectangle Tiling problem, Kenyon [6] (1996) continues to utilize the theory of resistor networks to give an upper bound on the number of tiles needed. It is proved that for any coprime p, q with $p > q$, a $p \times q$ rectangle can be tiled using less than $p/q + C \log(p)$ squares with integer side lengths, which can be easily extended to an upper bound for the Prime Tiling problem.

As for lower bounds, Conway [3] (1964) established a lower bound on the number of tiles for a Prime Tiling using the resistor approach yet again: a prime tiling of a $p \times q$ rectangle requires at least $\log_2(\max\{p, q\})$ tiles. Conway’s result has been recently improved by D. Fomin [5], who proved the lower bound $\log_\tau(\max\{p, q\})$, where $\tau \approx 1.8637 \dots$

However, as noted at the end of both [1] and [6], the very natural higher dimensional, or even 3-dimensional, analogues of the above two problems of tiling hypercuboids with hypercubes are very difficult to attack. This is largely due to the fact that “there is no satisfactory analogue [of resistor networks] in three dimensions,” as stated in [1]. One of the best results in higher dimensions so far is found in [9] by M. Walters in 2009, who proves an upper bound of $(\log p)^{O(d)}$ and a lower bound of $\Omega((\log p)^{d/2})$ for the d -dimensional analogue of the Coprime Rectangle Tiling problem where p denotes the longest side length. While this upper bound could be worked into an upper bound in the higher dimensional Prime Tiling problem, the lower bound simply does not extend to the higher dimensional Prime Tiling problem as we shall see later in this introduction in a counterexample of a prime tiling of hypercubes whose number of tiles is linear in $\log(p)$. One will also notice that the lower bound in [9] is obtained by taking a 2-dimensional cross section of the hypercube tiles, circumventing but not resolving the issue noticed above in [1] and [6].

Our goal is to give lower bounds to the Prime Tiling problem, i.e. *Mrs Perkins’s quilt*, both in 2-dimensions and higher dimensions, as well as lower bounds for natural related questions about some more general tilings. To achieve this, we first rephrase the problem at hand, switching to a completely new perspective, and introduce novel techniques to the problem to avoid the known difficulties.

We formally state the Prime Tiling problem, i.e. *Mrs Perkins’s quilt* here again:

- (1) *What is the minimal number of square tiles with integer side lengths required to tile a $p \times q$ rectangle, given that the lengths of the square tiles are jointly coprime?*

While this is a perfectly valid and natural problem, we study an equivalent one that is phrased quite differently:

- (2) *Suppose a rectangle is tiled by n squares. How large is the minimal-sized scaling of the rectangle such that all of its square tiles have integer side lengths?*

These are equivalent questions in the sense that, a lower bound for the minimal number of square tiles required is equivalent to an upper bound on the size of the scaling. For example, the above mentioned $\log_2(\max\{p, q\})$ estimate of Conway in [3] is equivalent to the statement that if a rectangle is tiled by n squares then there is a scaling of it with longest side length at most 2^n such that all square tiles have integer side lengths. In fact, (2) can even be considered as a quantitative strong Dehn’s Theorem, while estimates for (1) only imply estimates for (2) by using strong Dehn’s theorem.

The difference in the perspectives of these equivalent problems allows us to think outside of the box and apply new techniques that otherwise would never be considered. In our paper, we develop such a technique through a brand new approach to this classical problem, and apply it in various settings, demonstrating its universality in dealing with such tiling problems. In the classical case of *Mrs Perkins’s quilt* we prove the following:

Theorem 1.2. *For any tiling of a rectangle by n squares, one can find a scaling of it with longest side length at most 4^n such that all squares have integer side lengths.*

This result translates to a lower bound of $\frac{1}{2} \log_2(\max\{p, q\})$ in the Prime Tiling problem. Although the new machinery gives weaker results in this case compared to Conway and Fomin, we shall see that it is able to break many barriers which the classical method has thus far failed to overcome.

In order to prove Theorem 1.2, we demonstrate the following result, which seems to be interesting in itself:

Theorem 1.3. *Suppose a rectangle is tiled by squares and both x -coordinates or both y -coordinates of the vertices of the rectangle are integers. If each coordinate of every vertex of each square tile has distance less than $\frac{1}{4}$ to an integer, then all square tiles have integer side lengths.*

In order to use Theorem 1.3 to obtain Theorem 1.2, we will use the Simultaneous Version of Dirichlet’s Approximation Theorem (Theorem 2.3) to re-scale the tiled rectangle and ensure that all coordinates of all tiling squares have distance less than $\frac{1}{4}$ to an integer. To optimize the scaling factor as much as we can, we provide a sharp estimate (Lemma 4.1) for the number of distinct coordinates in a rectangle tiling.

Although we prove Theorem 1.3 from scratch, the proof was partly inspired by the resistor network technique mentioned above (see more explanation before Lemma 3.1).

In fact, the previously mentioned result of Fomin [5] is more general: he proved that if a rectangle of size $p \times q$ is tiled by smaller rectangles of side length ratio p_i/q_i with $(p_i, q_i) = 1$ ($i = 1, \dots, n$) such that the side lengths of the rectangle tiles are jointly coprime then

$$\max\{p, q\} \leq \prod_{i=1}^n \tau^{\max\{p_i, q_i\}}, \text{ where } \tau \approx 1.8637.$$

With our approach, we prove the following result which is weaker than the above result of Fomin when $\max\{p_i, q_i\} \leq 6$ for most i , but much stronger otherwise since $8x < \tau^x$ for $x \geq 7$.

Theorem 1.4. *Given a tiling of a rectangle by n rectangles with side length ratios p_i/q_i for $p_i, q_i \in \mathbb{Z}$ ($i = 1, \dots, n$), one can find a scaling of it with longest side length at most*

$$8^n \prod_{i=1}^n \max\{p_i, q_i\}$$

such that all tiling rectangles have integer side lengths.

As for the higher dimensional analogue of the prime tiling problem, the lack of a higher-dimensional analogue of the resistor network technique made it hard for R. L. Brooks et al. [1] and Richard Kenyon [6] to generalize their results as mentioned above. On the contrary, the Diophantine approximation technique offers a simple solution: as long as we scale the tiling such that a 2-dimensional section of the tiling has only squares with integer side lengths, all hypercubes have integer side lengths. This way we will be able to prove the following:

Theorem 1.5. *For any tiling of a d -dimensional hypercuboid by n hypercubes, one can find a scaling of it with the shortest side length at most $4^{\frac{2(n-1)}{d}+1}$ and longest side length at most $n \cdot 4^{\frac{2(n-1)}{d}+1}$ such that all tiling hypercubes have integer side lengths.*

This roughly translates to a lower bound in the classical higher dimensional Prime Tiling problem of $\frac{d}{4} \log_2(p)$, which is not only a result in uncharted territory, it is also linear in $\log(p)$, which is tight up to a factor not depending on p as we will see in an example later.

Finally, we study tilings with equilateral triangles, focusing on parallelograms and equilateral triangles tiled by equilateral triangles. Three analogous problems arise in these types of tilings:

- (1) [Coprime Parallelogram Tiling] What is the minimal number of equilateral triangle tiles of integer side lengths needed to tile a parallelogram (whose interior angles are 60° or 120°) with side lengths p and q , where $p, q \in \mathbb{Z}$ and $(p, q) = 1$?
- (2) [Parallelogram Prime Tiling] What is the minimal number of equilateral triangle tiles of jointly coprime (integer) side lengths needed to tile a parallelogram (whose interior angles are 60° or 120°) with side lengths p and q , where $p, q \in \mathbb{Z}$?
- (3) [Triangle Prime Tiling] What is the minimal number of equilateral triangle tiles of jointly coprime (integer) side lengths needed to tile an equilateral triangle with side length $p \in \mathbb{Z}$?

By making use of the theory presented in [8] (specifically the equations (24) and (25) of page 471), one can apply the same techniques offered by [6] and [3] (with the Leaky Electrical Network defined in [2] and its respective matrix) obtaining a lower bound of $n \geq \log_2(p)$ for Triangle Prime Tiling and $n \geq \log_2(\max\{p, q\})$ for the parallelogram problems where n is the number of tiles used in the triangulation. Our estimates, which we present in the next theorem, are slightly worse than these lower bounds, however, we offer them here as it is a natural generalization of our technique in rectangular problems.

Theorem 1.6. *For any tiling of an equilateral triangle by n equilateral triangles, one can find a scaling of it with side length at most $4^{\frac{2n-2}{3}}$ such that all triangle tiles have integer side lengths.*

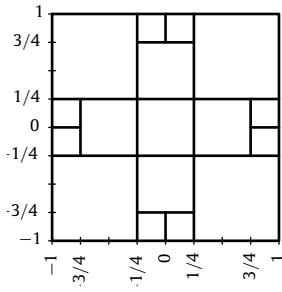


Fig. 1. Example to witness the sharpness of Theorem 1.3.

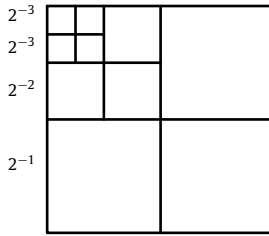


Fig. 2. An easy to generalize construction related to Theorem 1.2.

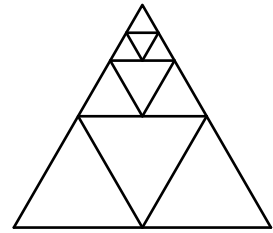


Fig. 3. Example to witness some sharpness of Theorem 1.6.

As before, this theorem gives a lower bound on the Prime Tiling problem of the form $n \geq \frac{3}{4} \log_2(p)$, where $p \in \mathbb{Z}$ is the side length of the triangle.

How sharp are the above results? Theorem 1.3 is sharp in the sense that the constant $\frac{1}{4}$ cannot be improved, we cannot even allow equality: one can decompose the square $[-1, 1] \times [-1, 1]$ into squares of side-lengths $3/4, 1/2$ and $1/4$ such that the coordinates of the vertices of the square tiles are $0, \pm 1/4, \pm 3/4$ and ± 1 (see Fig. 1).

The other results are sharp only in the sense that the exponential estimates are necessary. For Theorem 1.2, this is witnessed by the classical Fibonacci tiling: by induction a rectangle of size $u_n \times u_{n+1}$ can be tiled by n squares of side-lengths u_1, \dots, u_n , where u_n is the n -th Fibonacci number, which shows that 4^n cannot be replaced by anything better than $u_{n+1} \approx ((\sqrt{5} + 1)/2)^{n+1} / \sqrt{5}$ in Theorem 1.2. To get another construction, which is easier to generalize, we can trivially tile a unit square, for any positive integer k , into 3 squares of side length 2^{-i} , for $i = 1, \dots, k - 1$, and $3 + 1$ squares of side length 2^{-k} (see Fig. 2). This way we have $n = 3k + 1$ tiling squares, and the side length of the smallest one is $2^{-k} = 2^{-\frac{n-1}{3}}$. By replacing squares by equilateral triangles in the previous construction (see Fig. 3), we get a triangular tiling which shows that in Theorem 1.6 one cannot replace $4^{\frac{2n-2}{3}}$ by anything better than $2^{\frac{n-1}{3}}$. Finally, by replacing 3 by $2^d - 1$ and squares by hypercubes in the square tiling, of which Fig. 2 is an example, we get a hypercube tiling which shows that in Theorem 1.5 one cannot replace $4^{\frac{2(n-1)}{d} + 1}$ by anything better than $2^{\frac{n-1}{2^d - 1}}$. In terms of the classical setting of Prime Tiling, this example tiles a d -dimensional hypercuboid with side length $p = 2^r$ by $(2^d - 1)r + 1 = (2^d - 1)\lceil \log_2(p) \rceil + 1$ tiles, linear in $\log(p)$ as promised.

It is also worth noting, and easy to prove, that any tiling by rectangles of an axis-parallel rectangle in the Cartesian plane has axis-parallel rectangular tiles. So, from now on we will always assume that these tiles have axis-parallel sides. Something similar happens when dealing with equilateral triangle tilings, as the sides of the tiles will also be parallel to the respective sides of the tiled figure.

The paper is organized as follows. First, to illustrate and introduce some of our tools and techniques, we give a new simple proof of Max Dehn’s theorem in Section 2 using simultaneous Diophantine approximation. In Section 3, we prove Theorem 1.3. In Section 4, we prove the estimates about the number of distinct coordinates for rectangle tilings, hypercuboid tilings and for some triangle tilings and we prove Theorem 1.2. Section 5 proves our results about tilings with rectangles, Section 6 is about the higher dimensional generalizations, and Section 7 is about tilings with triangles.

2. A new proof of max Dehn’s theorem using Diophantine approximation

Notation 1. For $x \in \mathbb{R}$, we denote the distance between x and the closest integer to x by $\|x\|$.

Definition 2.1. An important tool of this paper is the grid of $(\frac{1}{2}, \frac{1}{2})$ -translates of the integer grid lines. The resulting grid is composed of lines of the form $\{k + \frac{1}{2}\} \times \mathbb{R}$ ($k \in \mathbb{Z}$), which we call $\frac{1}{2}$ -shifted vertical grid lines, and by lines of the form $\mathbb{R} \times \{k + \frac{1}{2}\}$, which we call $\frac{1}{2}$ -shifted horizontal grid lines. Here, we follow the convention that the x -axis is horizontal and is directed to the right, and the y -axis is vertical and directed upwards.

Max Dehn’s theorem and Diophantine approximation are connected via the following two observations.

Observation 2.1. Let $R = [a, b] \times [c, d]$ be a rectangle tiled by axis parallel squares. Suppose that

- (i) at least one $\frac{1}{2}$ -shifted vertical grid line intersects R ,
- (ii) the $\frac{1}{2}$ -shifted vertical and horizontal grid lines do not contain any vertex of any square tile,
- (iii) there are equal numbers of $\frac{1}{2}$ -shifted vertical and $\frac{1}{2}$ -shifted horizontal lines through each square.

Then the aspect ratio of R is rational.

Proof. Let h and v be the number of $\frac{1}{2}$ -shifted horizontal and $\frac{1}{2}$ -shifted vertical grid lines that intersect R , respectively. By adding up the lengths of the intersections of the $\frac{1}{2}$ -shifted vertical grid lines and the squares, we get $v(d - c)$ and by adding up the lengths of the intersections of the $\frac{1}{2}$ -shifted horizontal grid lines and the squares, we get $h(b - a)$. By (iii) and because the sides of a square are equal, this implies that $v(d - c) = h(b - a)$, and therefore $\frac{d-c}{b-a} = \frac{h}{v} \in \mathbb{Q}$. \square

Observation 2.2. If $\|a_i\| < \frac{1}{4}$ for $i = 1, 2, 3, 4$ and $a_2 - a_1 = a_4 - a_3 > 0$, then there are equal numbers of $\frac{1}{2}$ -shifted vertical and $\frac{1}{2}$ -shifted horizontal lines through the square $[a_1, a_2] \times [a_3, a_4]$.

Proof. As $\|a_i\| < \frac{1}{4}$, we can write $a_i = A_i + \varepsilon_i$ uniquely for some $A_i \in \mathbb{Z}$ and $\varepsilon_i \in (-\frac{1}{4}, \frac{1}{4})$. The number of $\frac{1}{2}$ -shifted horizontal and $\frac{1}{2}$ -shifted vertical grid lines through the square are $A_4 - A_3$ and $A_2 - A_1$, respectively. Therefore, we only need to show $A_2 - A_1 = A_4 - A_3$.

Since we have a square and $a_2 - a_1 = a_4 - a_3$,

$$(A_2 - A_1) + (\varepsilon_2 - \varepsilon_1) = (A_4 - A_3) + (\varepsilon_4 - \varepsilon_3)$$

$$\implies (\varepsilon_2 - \varepsilon_1) - (\varepsilon_4 - \varepsilon_3) = (A_4 - A_3) - (A_2 - A_1).$$

However, the left-hand side is within $(-1, 1)$ and the right-hand side is an integer. Hence, both sides of the equation are exactly 0, which completes the proof. \square

Proof of Theorem 1.1. Therefore, by the above observations, in order to prove that the aspect ratio of a rectangle with square tiling is rational all we need to do is to re-scale the rectangle with the tiling such that both coordinates of every vertex of each square tile have distance less than $\frac{1}{4}$ to integers and the side-lengths of the rectangle are at least 1. This is achieved by a scaling making the sides of the rectangle become at least 1 first and the scaling given by Theorem 2.3 afterwards. \square

The existence of such a scaling is clearly guaranteed by the following well known theorem. To make the argument self-contained and because the proof is short and elegant, we include it.

Theorem 2.3 (Dirichlet's Approximation Theorem, Simultaneous Version). Given real numbers $\{a_1, \dots, a_k\}$ and a natural number $N > 0$, there exists $q \in \mathbb{Z}$, $1 \leq q \leq N^k$ such that $\|qa_i\| < \frac{1}{N}$ for all i .

Proof. The proof is a simple application of the pigeonhole principle. Consider the cube $[0, 1)^k \subset \mathbb{R}^k$, which is the union of the N^k disjoint cubes of the form $[\frac{i_1}{N}, \frac{i_1+1}{N}) \times [\frac{i_2}{N}, \frac{i_2+1}{N}) \times \dots \times [\frac{i_k}{N}, \frac{i_k+1}{N})$ for $(i_1, \dots, i_k) \in \{0, \dots, N-1\}^k$. Take the $N^k + 1$ points $(\{Ma_1\}, \dots, \{Ma_k\}) \in [0, 1)^k$ for $0 \leq M \leq N^k$ where $\{x\}$ denotes the fractional part of x . By the pigeonhole principle, there are two points $(\{M_1a_1\}, \dots, \{M_1a_k\})$ and $(\{M_2a_1\}, \dots, \{M_2a_k\})$ with $0 \leq M_1 < M_2 \leq N^k$ which are both elements of the same cube. Then taking $q = M_2 - M_1$ suffices. \square

3. Square tilings with almost integer side-lengths must have integer side-lengths

In this section, we exploit a very special property of the set of coordinates involved in a square tiling. Roughly speaking, if all coordinates of each square are close enough to integers, then that necessitates, with a few extra conditions, all coordinates to be exactly integers.

The method used in this section, especially Lemma 3.1, may seem to be pulled out of thin air. However, it is inspired by the correspondence between resistor networks and tilings of rectangles: the side lengths of the tiles naturally produce a solution to Kirchoff's equations (see [7]). We make an unexpected application of the uniqueness of this solution by using $\frac{1}{2}$ -shifted horizontal and vertical grid lines to produce another solution to obtain certain equalities. Now Lemma 3.1 is a proof of the uniqueness statement put into the tiling context, while Lemma 3.2 is its application. We shall use Lemma 3.1 for the function $r(x) = \lfloor x + 1/2 \rfloor$, and in that case, (iii) of Observation 2.1 will guarantee that (1) of Lemma 3.1 indeed holds.

Lemma 3.1. Suppose that a rectangle $[a, b] \times [c, d]$ is tiled by rectangles $R_i = [a_i, b_i] \times [c_i, d_i]$ ($i = 1, \dots, n$) and we have a function $r : \mathbb{R} \rightarrow \mathbb{R}$ such that $r(c) = c$, $r(d) = d$ and

$$(b_i - a_i) \cdot (r(d_i) - r(c_i)) = (r(b_i) - r(a_i)) \cdot (d_i - c_i) \quad (i = 1, \dots, n). \tag{1}$$

Then for every i we have

$$r(c_i) = c_i, \quad r(d_i) = d_i.$$

Proof. Let $\delta(x) = r(x) - x$. By subtracting $(b_i - a_i) \cdot (d_i - c_i)$ from (1) we obtain

$$(b_i - a_i) \cdot (\delta(d_i) - \delta(c_i)) = (\delta(b_i) - \delta(a_i)) \cdot (d_i - c_i) \quad (i = 1, \dots, n). \tag{2}$$

Take the union of the horizontal edges of the tiling rectangles and let $S_j = [u_j, v_j] \times \{h_j\}$ ($j = 1, \dots, m$) be its connected components. We can clearly suppose that $h_1 = c$ and $h_m = d$. Thus the assumptions $r(c) = c$ and $r(d) = d$ imply that

$$\delta(h_1) = \delta(h_m) = 0. \tag{3}$$

Using (2) first and then (3), we obtain

$$\begin{aligned} \sum_{i=1}^n \frac{b_i - a_i}{d_i - c_i} \cdot (\delta(d_i) - \delta(c_i))^2 &= \sum_{i=1}^n (\delta(b_i) - \delta(a_i)) \cdot (\delta(d_i) - \delta(c_i)) \\ &= \sum_{j=2}^{m-1} \left(\sum_{i:d_i=h_j} (\delta(b_i) - \delta(a_i)) \cdot \delta(h_j) - \sum_{i:c_i=h_j} (\delta(b_i) - \delta(a_i)) \cdot \delta(h_j) \right). \end{aligned}$$

Since for any $j = 2, \dots, m - 1$ the segment $[u_j, v_j] \times \{h_j\}$ can be obtained both as the union of upper and lower edges of the rectangles of the tiling, we also have

$$\delta(v_j) - \delta(u_j) = \sum_{i:d_i=h_j} (\delta(b_i) - \delta(a_i)) = \sum_{i:c_i=h_j} (\delta(b_i) - \delta(a_i)).$$

Thus the previous sum over j is zero, which implies that $\delta(d_i) = \delta(c_i)$ for every i . Notice that $\delta(c) = 0$ and $\delta(c_i) = \sum_{j \in S} (\delta(d_j) - \delta(c_j)) + \delta(c)$ for some $S \subseteq \{1, \dots, n\}$ with $c_j < d_j \leq c_i$ for each $j \in S$ due to the connectivity, hence $\delta(d_i) = \delta(c_i) = 0$ for all i , which completes the proof. \square

By applying Lemma 3.1 for $r(x) = \lfloor x + \frac{1}{2} \rfloor$ (in other words, the closest integer to x) we get the following.

Lemma 3.2. Consider a rectangle tiled by squares. Suppose that the conditions (i)-(iii) of Observation 2.1 are satisfied, and the vertices of the rectangle have integer y -coordinates. Then each square has side length equal to the number of $\frac{1}{2}$ -shifted horizontal lines through it. In particular, all squares have integer side lengths.

Proof. Let $r(x) = \lfloor x + \frac{1}{2} \rfloor$. We claim that the conditions in Lemma 3.1 are satisfied. Indeed, as $c, d \in \mathbb{Z}$, $r(c) = c$ and $r(d) = d$ by (ii). For each i , $b_i - a_i = c_i - d_i$ is the side length of the square; $r(b_i) - r(a_i)$ is the number of $\frac{1}{2}$ -shifted vertical grid lines through the square, while $r(d_i) - r(c_i)$ is the number of $\frac{1}{2}$ -shifted horizontal grid lines through the square by (iii). Hence $(c_i - d_i)(r(b_i) - r(a_i)) = (b_i - a_i)(r(d_i) - r(c_i))$. Applying Lemma 3.1 we get $d_i - c_i = r(d_i) - r(c_i)$ for each i , which completes the proof. \square

Proof of Theorem 1.3. Combining Lemma 3.2 with Observation 2.2, we clearly obtain Theorem 1.3. \square

4. Bounding number of distinct coordinate values and the proof of Theorem 1.2

To use Theorem 1.3 we need to re-scale the tiled rectangle, by using the Simultaneous Version of Dirichlet’s Approximation Theorem (Theorem 2.3), to make all coordinates of all the vertices of tiling squares closer than $\frac{1}{4}$ to an integer. How many coordinates do we have to work with? If we have n square tiles then, trivially, we cannot have more than $4n$ coordinates, and if we also notice that most coordinates are counted at least twice we get the immediate $2n + 2$ estimate. To optimize the scaling factor as much as possible, we prove the following sharp estimate for the number of distinct coordinates in a rectangle tiling. (To check that $n + 3$ is indeed sharp is left to the reader as an exercise.)

Lemma 4.1. Suppose that a rectangle $R = [a, b] \times [c, d]$ is tiled by rectangles $R_i = [a_i, b_i] \times [c_i, d_i]$ ($i = 1, \dots, n$). Then the number of distinct x -coordinates plus the number of distinct y -coordinates of the vertices of the tiling rectangles R_1, \dots, R_n is at most $n + 3$; that is,

$$\#\{a_1, \dots, a_n, b_1, \dots, b_n\} + \#\{c_1, \dots, c_n, d_1, \dots, d_n\} \leq n + 3. \tag{4}$$

Now we show how Lemma 4.1 completes the proof of Theorem 1.2, which we repeat here for convenience, and then we prove Lemma 4.1 in a more general form.

Theorem 4.2. For any tiling of a rectangle by n squares, one can find a scaling of it with the longest side length at most 4^n such that all squares have integer side lengths.

Proof modulo Lemma 4.1. We let the initial rectangle to have bottom left corner at coordinate $(0, 0)$ and top left corner $(0, 1)$, and let $(0, 0)$ to $(0, 1)$ be the longest side of the rectangle. Now consider the set

$$S := \{x|(x, y) \text{ or } (y, x) \text{ is a vertex of a square tile for some } y\} \setminus \{0, 1\}.$$

From Lemma 4.1, $|S| \leq (n + 3) - 3 = n$, where we take away 3 coordinates because we exclude $\{0, 1\}$ and in the left-hand side of (4) the number 0 appears in both terms and 1 appears in the second term since $(0, 0)$ and $(0, 1)$ are vertices of the rectangle.

Applying Theorem 2.3 on the set of real numbers S with $N = 4$, we can find $q \in \mathbb{Z}$ such that $1 \leq q \leq 4^{|S|}$ and $\|qx\| < \frac{1}{4}$ for all $x \in S$. Scale the tiling by a factor of q , then every vertex of each square has its coordinates in the form of qx for some x in S , and hence are closer than $\frac{1}{4}$ to an integer. Also, the rectangle has bottom side coordinate 0 and top side coordinate q , which are both integers. Therefore by Theorem 1.3, all squares have integer side length.

Now, the longest sides of the rectangle have size $q \leq 4^{|S|} \leq 4^n$. \square

Remarks. 1. Without Lemma 4.1, using the trivial $2n + 2$ estimate in (4) instead of $n + 3$, we would get this theorem with the slightly worse 4^{2n-1} estimate instead of 4^n .

2. Diophantine approximation is very carefully tailored to this situation. Not only does it very naturally guarantee the coordinates to be closer than $\frac{1}{4}$ to integers, but the fact that q is an integer also helps to ensure both x -coordinates or both y -coordinates of the vertices of the scaled rectangle are integers.

Instead of proving Lemma 4.1 directly, we prove two generalizations of it. The first one (Lemma 4.3) will be needed for the higher dimensional variations, while the second one (Lemma 4.4) for the variants with triangle tilings, and either of them clearly implies Lemma 4.1.

Lemma 4.3. *Suppose that a d -dimensional axis-parallel hypercuboid C is tiled by n hypercuboids $\{C_1, \dots, C_n\}$. Then*

- (i) *the intersection of the interior of C and the union of the boundaries of C_1, \dots, C_n can be covered by $n - 1$ hyperplanes such that each hyperplane is perpendicular to one of the coordinate axes, and*
- (ii) *there exist distinct $i, j \in \{1, \dots, d\}$ such that the number of distinct i -th coordinates plus the number of distinct j -th coordinates of the vertices of the hypercuboids $\{C_1, \dots, C_n\}$ is at most*

$$\frac{2(n - 1)}{d} + 4.$$

Proof. It is easy to see that the (i) implies (ii), so it is enough to prove (i). Fix one of the d axis directions, say $i \in \{1, \dots, d\}$, and consider the union of the $2n(d - 1)$ -dimensional faces of the C_j 's that are perpendicular to the given direction. Let T_i be the collection of connected components of this union. Define T'_i to be T_i , but without the top and bottom $(d - 1)$ -dimensional faces of C and define $T = \bigsqcup_{i=1}^d T'_i$, where here and in the sequel “top” indicates the largest i -th coordinate and “bottom” indicates the smallest i -th coordinate. Clearly, it is enough to prove that the set T contains at most $n - 1$ elements.

We consider the natural partial ordering on the points of \mathbb{R}^d , that is: we say that $p \leq p'$ if $p_i \leq p'_i$ for each coordinate of p and p' . Additionally, p is a *minimal point* of $A \subset \mathbb{R}^d$ if $p \in A$ and there does not exist $p' \in A$ such that $p' < p$. Clearly, any compact subset of \mathbb{R}^d has at least one minimal point and any compact axis-parallel hypercuboid has a unique minimal point, which we call its *bottom-left vertex*.

For each $S \in T$ we choose a minimal point p of S . Since T does not contain any top face of C this point p must be also a minimal point of a tiling cuboid C_j . Also, as T does not contain any bottom face of C , this C_j cannot be the cuboid tile that contains the bottom-left vertex of C . Therefore, to prove that T indeed has at most $n - 1$ elements, and thus to complete the proof of the lemma, it is enough to show that distinct $S, S' \in T$ cannot have a common minimal point.

So, suppose that $p = (p_1, \dots, p_d)$ is a common minimal point of these distinct S, S' . Choose i and i' such that $S \in T'_i$ and $S' \in T'_{i'}$. Clearly, $i \neq i'$. Due to the fact that $S \in T'_i$, we have that for small enough ε the open line segment s between p and $p - \varepsilon e_i$, where e_i denotes the unit vector in the direction of the i -th axis, is contained in C_j for some j , see Fig. 4. As s is contained in the hyperplane of S' , the minimality of p in S' implies that s must be in the interior of C_j . Furthermore, this implies that if we write $C_j = [u_1, v_1] \times \dots \times [u_d, v_d]$, then we must have $v_i = p_i$. But then the top $(d - 1)$ -dimensional face of C_j perpendicular to the i -th axis must be contained in S and so p is not minimal in S , which is a contradiction. \square

Lemma 4.4. *Suppose that T is tiled by tiles T_1, \dots, T_n , where T and each T_i is a trapezoid with two horizontal sides or a triangle with a horizontal side. Then the intersection of the interior of T and the union H of the boundaries of T_1, \dots, T_n can be covered by $n - 1$ lines.*

Proof. Let L be the set of lines that contains at least one of the line segments of H . We need to show that $\#L \leq n - 1$. To this end, we define an injective, but not surjective, function $f : L \rightarrow \{T_1, \dots, T_n\}$.

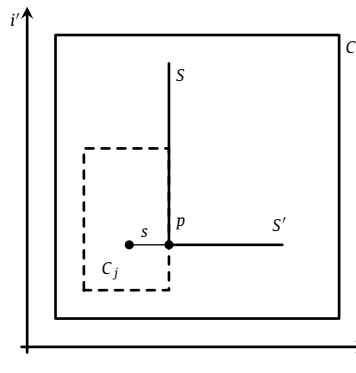


Fig. 4. Cross section through p .

If $\ell \in L$ is horizontal, then consider those tiles whose boundaries intersect ℓ and are contained in the closed half-plane above ℓ . It is easy to see that there exists at least one such tile and there is a clear ordering of these tiles from left to right, so, we can define $f(\ell)$ as the leftmost one. If $\ell \in L$ is non-horizontal then consider the tiles whose left edge is on ℓ and let $f(\ell)$ be the one with the lowest left side. Note that the bottom-left tile is not in the range of f since its domain L does not contain the left and bottom sides of T . Thus f is not surjective.

To prove that f is injective it is enough to check that we cannot have $f(h) = T_i = f(v)$ for any horizontal $h \in L$ and nonhorizontal $v \in L$. Suppose, to the contrary, that this is the case. Let $(a, b) = h \cap v$. This point must be the bottom-left vertex of T_i . Now observe that v is not the left side of T because H is contained in the interior of T , so no line segment of H is contained in the boundary of T . Thus, for small enough $\varepsilon > 0$ the line segment $s = (a - \varepsilon, a) \times \{b\}$ must be contained in some tile T_j . The segment s must be in the interior of T_j , otherwise T_j would not be the leftmost tile whose boundary intersects h and is contained in the closed half-plane above h . Using that T_j is a trapezoid with two horizontal sides or a triangle with a horizontal side, this implies that (a, b) is in the interior of the right edge of T_j . Since T_i is the leftmost tile whose boundary intersects h and is contained in the closed half-plane above h , there cannot be a tile between the right edge of T_j and the left edge of T_i , so v contains the right edge of T_j . But then, on the other side of v there must be at least one tile lower than T_i , contradicting $f(v) = T_i$. \square

5. Tiling by rectangles

In this section, we consider tiling rectangles with smaller rectangles of pre-emptively fixed sizes and we prove Theorem 1.4 mentioned in the introduction. We work under the set up that a rectangle is tiled by n smaller rectangular tiles, where the i -th tile has ratio-of-sides p_i/q_i , where $p_i \in \mathbb{Z}$ corresponds to the vertical side and $q_i \in \mathbb{Z}$ to the horizontal side. We investigate whether there will be similar results as the case of square tiles.

In the same manner as for squares, we get the following more general form of Lemma 3.2 by again applying Lemma 3.1 for $r(x) = \lfloor x + \frac{1}{2} \rfloor$.

Theorem 5.1. Consider a rectangle tiled by n rectangles. Let the i -th rectangle, R_i , have ratio of its vertical side to its horizontal side equal to p_i/q_i where $p_i, q_i \in \mathbb{Z}$. Suppose that

- (i) the vertices of the tiled rectangle have integer y -coordinates;
- (ii) there is no vertex of any rectangular tile having y -coordinate with fractional part $\frac{1}{2}$; and
- (iii) the ratio of the number of $\frac{1}{2}$ -shifted horizontal and $\frac{1}{2}$ -shifted vertical grid lines through the i th tile is p_i/q_i or both numbers vanish.

Then each rectangle has horizontal side length equal to the number of $\frac{1}{2}$ -shifted vertical grid lines through it.

Using a slightly different Diophantine approximation from the square case, we arrive at the following analogue of Theorem 1.3.

Lemma 5.2. Given a tiling of a rectangle by n rectangles with the ratios of the vertical side to the horizontal side equal to $\frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{Z}$. Suppose the i th rectangle R_i has actual size $a_i \times b_i$, and

- the vertices of the tiled rectangle have integer y -coordinates;
- for each i , the ratio a_i/p_i is closer than $\frac{1}{2 \max\{p_i, q_i\}}$ to an integer;
- all coordinates of each vertex of the tiles are closer than $\frac{1}{4}$ to an integer.

Then the side lengths of the i th rectangular tile are integer multiples of (p_i, q_i) . In particular, the side lengths of all the rectangular tiles are integers.

Proof. Now consider the i th rectangle tile R_i , in particular the y -coordinates of its vertices. Write these two coordinates to be $r_1 + \varepsilon_1 < r_2 + \varepsilon_2$, where $r_j \in \mathbb{Z}$ and $\varepsilon_j \in (-\frac{1}{4}, +\frac{1}{4})$. Moreover let $a_i/p_i = b_i/q_i = s_i + \delta_i$ where $s_i \in \mathbb{Z}$ and $\delta_i \in (-\frac{1}{2\max\{p_i, q_i\}}, +\frac{1}{2\max\{p_i, q_i\}})$ by the hypotheses of the lemma. Now we have

$$\begin{aligned} (r_2 + \varepsilon_2) - (r_1 + \varepsilon_1) &= a_i = (s_i + \delta_i)p_i \\ \implies r_2 - r_1 - s_i p_i &= \delta_i p_i - \varepsilon_2 + \varepsilon_1 \end{aligned}$$

Now $LHS \in \mathbb{Z}$ and $RHS \in (-1, 1)$. Hence both sides are exactly 0. In particular $r_2 - r_1 = s_i p_i$. In other words, the number of $\frac{1}{2}$ -shifted horizontal grid lines through the tile R_i is $s_i p_i$. One could similarly show that the number of $\frac{1}{2}$ -shifted vertical grid lines through R_i is $s_i q_i$.

We are now ready to apply Theorem 5.1 to this tiling because all three conditions are satisfied. Therefore each rectangle tile R_i has horizontal side length equal to the number of $\frac{1}{2}$ -shifted vertical grid lines through it, $s_i q_i$. Then clearly the vertical side length is $s_i p_i$. \square

One can notice that Lemma 5.2 is exactly Theorem 1.3 in the case where tiles have dimension $(1 : 1)$, that is, they are squares.

Theorem 5.3 (Dirichlet's Approximation Theorem, Varied Simultaneous Version). *Given sets of real numbers S_0, \dots, S_r and natural numbers $m_0, \dots, m_r \geq 1$, there exists $q \in \mathbb{Z}$, with $1 \leq q \leq m_0^{|S_0|} \dots m_r^{|S_r|}$ such that $\|qa_i\| < \frac{1}{m_i}$ for any $a_i \in S_i$ and for all i .*

Proof. This can be proved analogously to Theorem 2.3. \square

We now have the conditions to prove Theorem 1.4, which we repeat here for the readers' comfort.

Theorem 5.4. *Given a tiling of a rectangle by n rectangles with ratios of side lengths p_i/q_i , $p_i, q_i \in \mathbb{Z}$ respectively, one can find a scaling of it with longest side length at most $8^n \prod_{i=1}^n \max\{p_i, q_i\}$ such that all rectangles have integer side lengths.*

Proof. Again, we place the rectangle such that the bottom left corner has coordinate $(0, 0)$, top left corner $(0, 1)$ and let $(0, 0)$ to $(0, 1)$ be the longest side of the rectangle. For the i th rectangle tile R_i , denote the vertical side-length by a_i and the horizontal one as b_i . Let

$$\begin{aligned} S_0 &:= \{x \mid (x, y) \text{ or } (y, x) \text{ is a vertex of a rectangle tile for some } y\} \setminus \{0, 1\}, \\ S_i &:= \{a_i/p_i\} \quad (i = 1, \dots, n). \end{aligned}$$

Now we apply Theorem 5.3 to these sets, $m_0 = 4$ and $m_i = 2 \max\{p_i, q_i\}$, to find $q \in \mathbb{Z}$, $1 \leq q \leq 4^{|S_0|} \prod_{i=1}^n ((2 \max\{p_i, q_i\})^{|S_i|})$ such that

$$\left\| q \frac{a_i}{p_i} \right\| < \frac{1}{2 \max\{p_i, q_i\}}, \text{ and } \|qx\| < \frac{1}{4} \text{ for all } x \in S_0.$$

Scale the entire tiling by q . Now apply Lemma 5.2 to the scaled tiling to obtain that all sides of all rectangles are of integer length.

The longest side of the rectangle is q , and we know $q \leq 4^{|S_0|} \prod_{i=1}^n (2 \max\{p_i, q_i\})$. By Lemma 4.1, $|S_0| \leq (n + 3) - 3 = n$. Therefore, $q \leq 8^n \prod_{i=1}^n \max\{p_i, q_i\}$. \square

Remark. An analog of Observation 2.2 does exist in this case, but the resulting bounds are worse. Therefore, we have bounded the distance from a_i/p_i to an integer to guarantee the number of $\frac{1}{2}$ -shifted vertical and $\frac{1}{2}$ -shifted horizontal grid lines are in desired proportions after scaling without the help of an analogous observation.

6. Tiling of hypercuboids

We start this subsection by proving Theorem 1.5. As before, we write it again for convenience.

Theorem 6.1. *For any tiling of a hypercuboid with n hypercubes, one can find a scaling of it with the shortest side length at most $4^{\frac{2(n-1)}{d} + 1}$ and longest side length at most $n \cdot 4^{\frac{2(n-1)}{d} + 1}$ such that all tiling hypercubes have integer side lengths.*

Proof. Let $C = \{C_1, \dots, C_n\}$ be the set of tiles. By Lemma 4.3 we know that there exist two distinct coordinates, say the i -th and j -th coordinates, such that the number of distinct i -th coordinates plus the number of distinct j -th coordinates of the vertices of the tiles is at most $\frac{2(n-1)}{d} + 4$. Without loss of generality we can assume that $i = 1$ and $j = 2$.

Now, we translate and scale the hypercuboid in question, H , so that it has vertices at $(0, \dots, 0)$ and $(1, 0, \dots, 0)$.

Let S be the set of the first and second coordinates of all the vertices. Since two of the four first and second coordinates of the vertices of H are both 0, we can conclude $S \setminus \{0, 1\}$ has at most $\frac{2(n-1)}{d} + 1$ elements. Therefore, applying Theorem 2.3 on $S \setminus \{0, 1\}$ with $N = 4$, we find $1 \leq q \leq 4^{\frac{2(n-1)}{d} + 1}$, $q \in \mathbb{Z}$ such that $\|qx\| < \frac{1}{4}$ for all $x \in S$.

Scale H by q . Then let P be any arbitrary 2-dimensional plane parallel to the first and second axes that does not pass through any vertices of the tiles, and whose intersection with H is nonempty. Consider the rectangle R dissected by the collection of nonempty squares among $C_i \cap P$'s. Applying Theorem 1.3 to this rectangle tiling implies all hypercubes involved in the section have integer side lengths. However, the choice of the section is arbitrary, so all hypercubes must have integer side lengths. The first coordinate side has length $q \leq 4^{\frac{2(n-1)}{d} + 1}$. To finish, note that every side of H is partitioned by sides of the tiles in C , so the longest side of H is at most n times the smallest side of H . Since we know that the first coordinate side of H is at most $4^{\frac{2(n-1)}{d} + 1}$, we can conclude that the shortest side length of H is at most $4^{\frac{2(n-1)}{d} + 1}$ and the longest side is at most $n \cdot 4^{\frac{2(n-1)}{d} + 1}$. \square

Remark. The estimate 4^n of Theorem 1.2 also works here: enumerating the coordinates so that the longest side of the hypercuboid is in the first coordinate, and using (i) of Lemma 4.3, we have that the number of the coordinates in any two directions is at most $n + 3$. A similar argument as in Theorem 6.1 gives a scaling for any hypercuboid tiled by n hypercubes, with the longest side length at most 4^n , such that all hypercubes have integer length. On the other hand, if $d > 2$, the 4^n bound is a better estimate than the one in Theorem 6.1 only for $n = 2$ and 3, in most cases it is much weaker.

Now we prove a higher dimensional generalization of Theorem 1.4.

Theorem 6.2. *Suppose there is a tiling of a hypercuboid by n hypercuboids where the k -th tile is similar to a hypercuboid $q_{k,1} \times \dots \times q_{k,d}$, where $q_{k,j} \in \mathbb{Z}$ for all $j = 1, \dots, d$. Let the longest side of the hypercuboid be along the i -th coordinate. Then, for any $j \neq i$ one can find a scaling of the tiled hypercuboid with the longest side length at most $8^n \prod_{k=1}^n \max\{q_{k,i}, q_{k,j}\}$ such that all tiles have integer side lengths.*

Proof. Let $C = \{C_1, \dots, C_n\}$ be the set of tiles. Without loss of generality the side $(0, \dots, 0)$ to $(1, 0, \dots, 0)$ is the longest of the hypercuboid, in particular, $i = 1$. Take any direction $j \neq 1$ and for the k th hypercuboid tile C_k , denote the first axis direction side length by a_k and the j th one as b_k . Then we have $\frac{a_k}{b_k} = \frac{q_{k,1}}{q_{k,j}}$. Let

$$S_0 := \text{Collection of all first and } j\text{-th coordinates of all vertices of the tiles } \setminus \{0, 1\},$$

$$S_k := \{a_k/q_{k,1}\} = \{b_k/q_{k,j}\} \quad (k = 1, \dots, n).$$

Now we apply Theorem 5.3 to these sets, to find $q \in \mathbb{Z}$, $1 \leq q \leq 4^{|S_0|} \prod_{k=1}^n (2 \max\{q_{k,1}, q_{k,j}\})$ such that

$$\left\| q \frac{a_k}{q_{k,1}} \right\| = \left\| q \frac{b_k}{q_{k,j}} \right\| < \frac{1}{2 \max\{q_{k,1}, q_{k,j}\}}, \text{ and } \|qx\| < \frac{1}{4} \text{ for all } x \in S_0.$$

Scale the entire tiling by q and let H be the tiled hypercuboid we obtain. As in Theorem 6.1, we take P , any arbitrary 2-dimensional plane parallel to the first and j -th axes that does not pass through any vertices, and whose intersection with H is nonempty. Consider the rectangle R dissected by the collection of nonempty rectangles among $C_i \cap P$'s. Then R together with its tiles satisfy the conditions of Lemma 5.2. Hence we conclude that the ordered pair of side lengths of $C_i \cap P$ is an integer multiple of $(q_{i,1}, q_{i,j})$. Therefore, all the side lengths of the hypercuboids are integers.

The longest side of H is q , and we know $q \leq 4^{|S_0|} \prod_{k=1}^n (2 \max\{q_{k,1}, q_{k,j}\})$. By Lemma 4.3, $|S_0| \leq n$. Therefore $q \leq 8^n \prod_{k=1}^n \max\{q_{k,1}, q_{k,j}\}$. \square

We could also have given an analogous bound to the longest side of H in terms of the smallest side, as in the final part of the proof of Theorem 6.1, but in this case the d , possibly different, side lengths of each tile would have given a more complex and artificial factor, rather than the elegant n factor found in Theorem 6.1. We left to the reader the task of obtaining this bound.

7. Triangle tilings

In this section, we study the dissections of an equilateral triangle or a trapezoid into equilateral triangles and we prove results similar to the ones obtained in previous sections. Here it will be more convenient to use an oblique coordinate

system with angle 60° between the axes. We use $(1, 0)$ and $(1/2, \sqrt{3}/2)$ for base vectors, and we will denote by $\Psi(a, b)$ the point with coordinates a and b in this oblique coordinate system; that is, we let

$$\Psi(a, b) = a(1, 0) + b \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) = \left(a + \frac{b}{2}, \frac{\sqrt{3}b}{2} \right).$$

We will refer from now on to the coordinates of this system as the “ Ψ -coordinates”. Let $T(a, b, c)$ denote the equilateral triangles with vertices of the form $\Psi(a, b), \Psi(a + c, b), \Psi(a, b + c)$, where $a, b, c \in \mathbb{R}, c \neq 0$. (Note that c can be negative.)

The proof of the following lemma was partly inspired by [2] in which “leaky” electric flows are assigned to triangle tilings of triangles and a uniqueness result is proved.

Lemma 7.1. *Suppose that an equilateral triangle $T(A, B, C)$ or a trapezoid with parallel sides on the horizontal lines $\Psi(\mathbb{R} \times \{B\})$ and $\Psi(\mathbb{R} \times \{B + C\})$ is tiled by triangles $T(a_i, b_i, c_i)$ ($i = 1, \dots, n$) and we have a function $r : \mathbb{R} \rightarrow \mathbb{R}$ such that $r(B) = B, r(B + C) = B + C$ and*

$$r(a_i + c_i) - r(a_i) = r(b_i + c_i) - r(b_i) \quad (i = 1, \dots, n). \tag{5}$$

Then for every i we have

$$r(b_i) = b_i, \quad r(b_i + c_i) = b_i + c_i.$$

Proof. Let $\delta(x) = r(x) - x$. By (5) we also have

$$\delta(a_i + c_i) - \delta(a_i) = \delta(b_i + c_i) - \delta(b_i) \quad (i = 1, \dots, n). \tag{6}$$

Let $S_j = \Psi([u_j, v_j] \times \{h_j\})$ ($j = 1, \dots, m$) be the connected components of the union of the horizontal edges and the vertices of the triangles of the tiling. We can clearly suppose that $B = h_1 \leq \dots \leq h_m = B + C$. Thus the assumptions $r(B) = B, r(B + C) = B + C$ imply that

$$\delta(h_1) = \delta(h_m) = 0. \tag{7}$$

Fix $j \in \{2, \dots, m - 1\}$ and consider the line segment $S_j = \Psi([u_j, v_j] \times \{h_j\})$. It contains vertices of the form $\Psi(a_i, b_i + c_i)$ (such that $b_i + c_i = h_j$) and horizontal sides of the form $\Psi([a_i, a_i + c_i] \times \{b_i\})$ or $\Psi([a_i + c_i, a_i] \times \{b_i\})$ (such that $b_i = h_j$) of some tiling triangles. Consider the triangles having one side contained in S_j . There are two options for a triangle, either it is completely located in the upper-half plane determined by the extension of S_j (that is of the form $\Psi([a_i, a_i + c_i] \times \{h_j\})$ with $c_i > 0$), or located in the lower-half plane of this extension (that is of the form $\Psi([a_i + c_i, a_i] \times \{h_j\})$ with $c_i < 0$). Each of these two sets gives a partition for S_j , so

$$\begin{aligned} \sum_{i: b_i = h_j, c_i > 0} (\delta(a_i + c_i) - \delta(a_i)) &= \delta(v_j) - \delta(u_j) = - \sum_{i: b_i = h_j, c_i < 0} (\delta(a_i + c_i) - \delta(a_i)) \implies \\ \sum_{i: b_i = h_j} (\delta(a_i + c_i) - \delta(a_i)) &= 0. \end{aligned} \tag{8}$$

Since the vertices and horizontal sides are alternating as we go around the segment S_j , the numbers of vertices and sides bordering upon the fixed S_j ($j = 2, \dots, m - 1$) are equal. So, considering the horizontal sides of $T(a_i, b_i, c_i)$ and the vertices $\Psi(a_i, b_i + c_i)$ in a fixed S_j we have

$$\sum_{i: b_i = h_j} \delta^2(b_i) = \delta^2(h_j) \cdot \#\{\text{Sides in } S_j\} = \delta^2(h_j) \cdot \#\{\text{Vertices in } S_j\} = \sum_{i: b_i + c_i = h_j} \delta^2(b_i + c_i).$$

Thus, summing for all $j \in \{1, \dots, m\}$ and recalling that $\delta(h_1) = \delta(h_m) = 0$ we get

$$\sum_{i=1}^n \delta^2(b_i + c_i) = \sum_{i=1}^n \delta^2(b_i). \tag{9}$$

Using (9), (6), (7) and (8) we obtain

$$\begin{aligned} \sum_{i=1}^n (\delta(b_i + c_i) - \delta(b_i))^2 &= \sum_{i=1}^n (\delta^2(b_i + c_i) + \delta^2(b_i) - 2\delta(b_i + c_i)\delta(b_i)) \\ &= \sum_{i=1}^n (2\delta(b_i)^2 - 2\delta(b_i + c_i)\delta(b_i)) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n -2\delta(b_i)(\delta(b_i + c_i) - \delta(b_i)) \\
 &= \sum_{i=1}^n -2\delta(b_i)(\delta(a_i + c_i) - \delta(a_i)) \\
 &= \sum_{j=2}^{m-1} -2\delta(h_j) \sum_{i:b_i=h_j} (\delta(a_i + c_i) - \delta(a_i)) = 0.
 \end{aligned}$$

Therefore $\delta(b_i + c_i) = \delta(b_i)$ for every i . Consider the following graph. Let $V = \{b_1, \dots, b_n, b_1 + c_1, \dots, b_n + c_n\}$ be the vertex set and connect the numbers x and y by an edge if there exists i such that $x = b_i, y = b_i + c_i$. Since $\delta(b_i + c_i) = \delta(b_i)$ for every i , the function δ is constant on each connected component of the graph. On the other hand, by construction, $\min V = B$ and every $x \in V \setminus \{B\}$ has an adjacent vertex $y \in V$ such that $y < x$. By finiteness, repeating this we obtain a path from any vertex of the graph to B , which implies that the graph is connected, so $\delta(b_i + c_i) = \delta(b_i) = \delta(B)$ for every i . Since by assumption $\delta(B) = 0$, this completes the proof. \square

Before going to the next result notice that, unlike the previous sections, here we will not make use of the “ $\frac{1}{2}$ -shifted vertical grid lines”, presented in Definition 2.1. Instead, we will use the lines resulting after applying Ψ to them, that is of the form $\Psi(\{y + 1/2\} \times \mathbb{R})$, and, henceforth, we will denote them by “ $\frac{1}{2}$ -shifted 60° diagonal grid lines”. The $\frac{1}{2}$ -shifted horizontal grid lines remain unchanged by Ψ , so it is not necessary to rename them. Now, for $r(x) = \lfloor x + \frac{1}{2} \rfloor$ the previous lemma immediately gives the next result, analogous to Lemma 3.2.

Theorem 7.2. Consider an equilateral triangle, T , with two of its sides parallel to the basis vectors $\Psi(1, 0)$ and $\Psi(0, 1)$, tiled by equilateral triangles. Suppose that

- the top and bottom of T have integer second Ψ -coordinates;
- there are no vertices of any triangle tile having Ψ -coordinates with fractional part $\frac{1}{2}$; and
- there is an equal number of $\frac{1}{2}$ -shifted 60° diagonal and $\frac{1}{2}$ -shifted horizontal grid lines through the sides of each triangle tile.

Then each triangle tile has side length equal to the number of $\frac{1}{2}$ -shifted diagonal grid lines through it. In particular, all equilateral triangles have integer side length.

We next present the analogous version of Observation 2.2.

Observation 7.3. If $\|a_i\| < \frac{1}{4}$ for $i = 1, 2, 3, 4$ and $l = a_2 - a_1 = a_4 - a_3 > 0$, then there are equal numbers of $\frac{1}{2}$ -shifted 60° diagonal and $\frac{1}{2}$ -shifted horizontal grid lines through the triangle $T(a_1, a_3, l)$.

Combining Theorem 7.2 with Observation 7.3, we obtain the following result.

Corollary 7.4. Suppose an equilateral triangle with two of its sides parallel to the basis vectors $\Psi(1, 0)$ and $\Psi(0, 1)$ is tiled by equilateral triangles. If all Ψ -coordinates of every vertex of each triangular tile are closer than $\frac{1}{4}$ to an integer, then all triangular tiles have integer side lengths.

Next, we provide a proof for Theorem 1.6, which we state again for convenience. In contrast with the rectangular case, in a triangular tiling by equilateral triangles, we need a few more steps, as we will see, before we can obtain a scaling of it in which all of its tiles have integer side lengths.

Theorem 7.5. For any tiling of an equilateral triangle by n equilateral triangles, one can find a scaling of it with side length at most $4^{\frac{2n-2}{3}}$ such that all triangular tiles have integer side lengths.

Proof. Let T be our tiled triangle. Without loss of generality we can assume that T is $T(0, 0, 1)$. By Lemma 4.4, we know that there are at most $n - 1$ lines (which are parallel to one of the three following vectors: $\Psi(0, 1)$, $\Psi(1, 0)$ and $\Psi(-1, 1)$) containing the union of boundaries of the tiles in the interior of T . Considering the 0° , 120° and 240° rotations of T and the triangles tiling it, with respect to the center of T , we can assure, by the Pigeonhole Principle, that in one of these three configurations the total number of first and second Ψ -coordinates is less than or equal to $\frac{2n-2}{3}$. Let S be this set of Ψ -coordinates. We then replace the tiling with the corresponding tiling resulting from the rotation where this happens. Since the rotation is with respect to the center of T , the triangle T remains unchanged. Applying Theorem 2.3 on the set of real numbers S with $N = 4$, we can find $q \in \mathbb{Z}$ such that $1 \leq q \leq 4^{|S|} \leq 4^{\frac{2n-2}{3}}$ and $\|qx\| < \frac{1}{4}$ for all $x \in S$. Scale the tiling of

the triangle by a factor of q , then every vertex of each triangular tile has its Ψ -coordinates in the form qx for some x in S , and hence are closer than $\frac{1}{4}$ to an integer. Also, T has bottom side second Ψ -coordinate equal to 0 and top vertex second Ψ -coordinate q , both integers. Therefore by Corollary 7.4, all triangular tiles have their side lengths equal to an integer. \square

Analogous results for tilings of isosceles trapezoids and parallelograms follow easily.

Corollary 7.6. *For any isosceles trapezoid or parallelogram tiled by n equilateral triangles, one can find a scaling of it with the longest side length at most $4^{\frac{2n}{3}}$ for the trapezoid and $4^{\frac{2n+2}{3}}$ for the parallelogram, such that all triangular tiles have integer side lengths.*

Proof. By adding one more equilateral triangle tile, with side length equal to the smaller base of the trapezoid and joining it to this base, we create an equilateral triangle tiling with $n + 1$ triangular tiles. So, by making use of Theorem 7.5, we get the $4^{\frac{2n}{3}}$ bound. The parallelogram case is almost the same, but we need two more tiles instead of one in order to create an equilateral triangle tiling with $n + 2$ tiles, thus obtaining the $4^{\frac{2n+2}{3}}$ bound. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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