When is the modified von Koch snowflake non-self-intersecting?

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Abstract

We prove that the modified von Koch snowflake curve, which we get as a limit by starting from an equilateral triangle (or from a segment) and repeatedly replacing the middle portion c of each interval by the other two sides of an equilateral triangle, (and the corresponding von Koch snowflake domain) is non-self-intersecting if and only if $c < \frac{1}{2}$. This answers a question of M. van den Berg.

1 Introduction

Consider the following generalization of the classical triadic von Koch snowflake. Let $0 < c < 1$ be given. Starting from an equilateral triangle, repeatedly replace the middle portion c of each interval by the other two sides of an equilateral triangle (see Fig. 1.) We shall call the polygons we get *approximating polygons* and the limit curve will be called *(von Koch) c-snowflake curve.* The closure of the union of all the above equilateral triangles will be called *(von Koch) c-snowflake domain.* The curve we get if we start from a segment will be called *c-adic von Koch curve.* (In [3, Example 9.5] this is called "modified von Koch curve".)

Note that $c = 1/3$ gives the classical case. Note also that the c-adic von Koch curve is a self-similar set and the c-snowflake curve consists of three c-adic von Koch curves. If the c-snowflake curve is a simple curve (in other words it is *self-avoiding / non-self-intersecting*) then it is clearly the boundary of the c-snowflake domain.

We shall call the c-snowflake domain *non-self-intersecting* or *self-avoiding* if the parts of the c-snowflake domain which are built on different segments of an approximating polygon have at most one common point. One can check that for any given $0 < c < 1$ the following three statements are equivalent:

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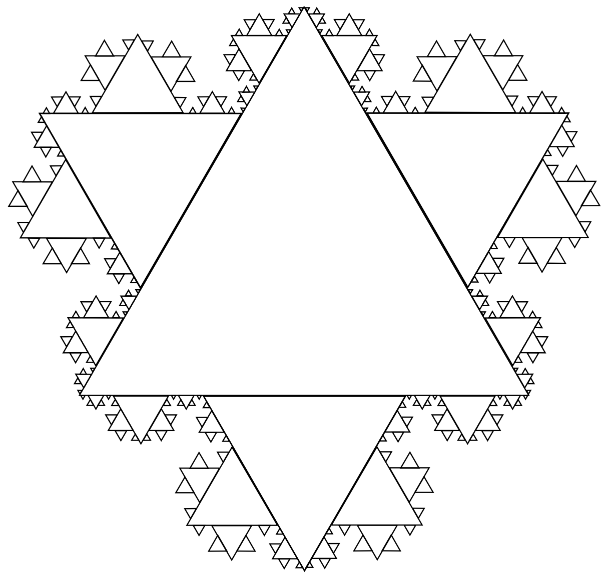

Figure 1: The c-snowflake for $c = 0.45$.

(1) the c-snowflake domain is non-self-intersecting, (2) the c-snowflake curve is non-self-intersecting (3) the c-adic von Koch curve is a simple curve.

M. van den Berg studied the heat equation for planar regions similar to the c-snowflake domain (see e.g. [1, 2]) and he posed at a colloquium talk at the University College London in 1998 the following natural question:

For what c is the c-snowflake domain non-self-intersecting?

The main result of the present paper is the following answer to this question.

Theorem 1 *The von Koch c-snowflake domain is non-self-intersecting if and only if* $c < \frac{1}{2}$. *Consequently the c-adic von Koch curve (and the c-snowflake curve) is a simple curve if and only if* $c < \frac{1}{2}$.

These fractals also appear in Mandelbrot's famous book [4]. Plate 56 in [4] is (a part of) a von Koch c-snowflake domain for which c is very close to $\frac{1}{2}$ (the Hausdorff dimension of the boundary is written to be $D \sim 1.4490$ there, while the dimension of the $\frac{1}{2}$-snowflake curve is $\log_2(1+\sqrt{3}) = 1.449984\ldots$) and the figure in [4] indeed shows that it is almost self-intersecting. This indicates that Mandelbrot probably already knew around 1980 that a c-snowflake is non-self-intersecting if $c < \frac{1}{2}$ (in fact, this is the much easier part of the above theorem) and that it is self-intersecting (in fact "self-contacting") for $c = \frac{1}{2}$. These facts already seem to imply the above theorem since intuitively it seems to be clear that if the c-snowflake is self-intersecting then for larger c it must be also self-intersecting. To show that this is not that clear (even though it is true in this case) we remark that if we replace equilateral triangles by regular

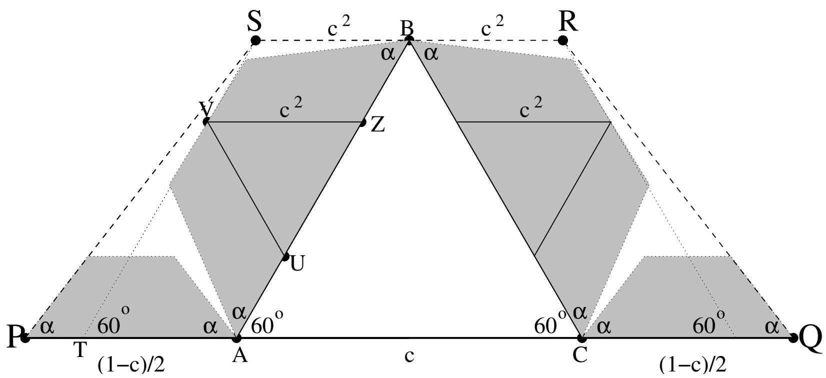

Figure 2: The separating trapezoids for $c = 0.45$.

k-gons everywhere in the definition then this would be false for some k. (This result will be published in a subsequent paper.)

2 Proof of Theorem 1

Proposition 2 *If $c < \frac{1}{2}$ then the c-snowflake domain is non-self-intersecting.*

Proof. Let PQ be a segment of one of the approximating polygons. Let $PABCQ$ be the part of the next approximating polygon that replaces PQ. (In other words, $PABCQ$ is the first generation descendant of PQ.) Let $AUVZB$ be the first generation descendant of AB. (See Fig. 2.) Let SR be a segment through B, parallel to PQ such that $SB = BR = VZ$.

We claim that (if $c < \frac{1}{2}$) the part of the c-snowflake domain that is built on the segment PQ is contained in the isosceles trapezoid $PQRS$. To show this we shall prove by induction that for any k the k-th generation of the descendents of PQ is contained in $PQRS$. By definition, this holds for $k = 1$. If $k > 1$ then applying the induction assumption for $k-1$ and for the segments PA, AB, BC and CQ we get 4 trapezoids (similar to $PQRS$) that contains the k-th generation of the descendents of PQ. So the only thing we have to check is that these 4 trapezoids are contained in $PQRS$. For the trapezoids on PA and CQ this is obvious. We can clearly suppose that PQ has unit length. Let T be the point on the half-line AP such that $TA = SB (= c^2)$. The trapezoid over AB is contained in the parallelogram $TABS$ and since $c < \frac{1}{2}$ implies $c^2 < (1-c)/2$ we have $TA < PA$, thus $TABS \subset PABS \subset PQRS$, therefore the trapezoid over AB is also contained in $PQRS$ and similarly so is the trapezoid over BC.

Since $TA < PA$ we get that the acute angle α of the (similar) isosceles trapezoids is less than $60°$. This implies that the trapezoids over PA, AB, BC and CQ are - except for the vertices A, B and C - pairwise disjoint. Thus the isosceles trapezoids we have constructed are separating the descendants of the different segments of each approximating polygon, which implies that the

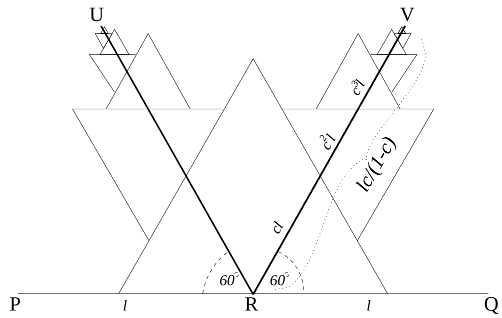

Figure 3: Lemma 4

c-snowflake domain is indeed non-self-intersecting. □

Remark 3 In [3] (Example 9.5) it is shown that if $0 < c \leq \frac{1}{3}$ then the Hausdorff dimension and the box dimension of the c-adic von Koch curve is the same as its similarity dimension (which is the solution of $2c^s + 2\left(\frac{1-c}{2}\right)^s = 1$) by observing that a triangle witnesses that the open set condition holds. The above argument (and Fig. 2) gives that the trapezoid $PQRS$ simply witnesses that the open set condition holds for any $0 < c \leq \frac{1}{2}$. Therefore the above cited statement in [3] can be extended to c-adic von Koch curves with $0 < c \leq \frac{1}{2}$.

Lemma 4 *If a segment PQ with length $2l$ is a segment of one of the approximating polygons of the c-snowflake domain K then the two closed segments (RU and RV in Fig. 3) with length $\frac{cl}{1-c}$ from the midpoint (R) of PQ at angle $60°$ on the outer side of PQ are also in K.*

Proof. Let build just a part of K as in Fig. 3. At each step we take just one new triangle, once on the right, once on the left. These triangles have sides $2cl, 2c^2l, 2c^3l, \ldots$, so the sum of the lengths of their median segments is $cl + cl^2 + \ldots = \frac{cl}{1-c}$. Thus RV is exactly the non-overlapping union of the median segments of these triangles, so RV (and similarly RU) is in K. □

We will use a coordinate system with angle $60°$. Let OXY be an equilateral triangle with side 1. We will say that P has coordinates (x, y) (or shortly $P = (x, y)$) if $\vec{OP} = x\vec{OX} + y\vec{OY}$ (see Fig. 4). We will denote by P^x and P^y the x and y coordinates of P in this coordinate system.

We will build a part of a c-snowflake domain K starting from the segment OT, where $T = (4/c, 0)$ (see Fig. 4). The left and the top vertices of the equilateral triangle with side $c\frac{4}{c} = 4$ that we put on OT are A and B, respectively. The lowest vertex of the triangle with side $4c$ that we put on AB is D, the midpoint of its left side is E. Thus $AD = \frac{4-4c}{2} = 2(1-c)$ and $DE = 4c/2 = 2c$. Choose G such that the angle DEG is $60°$ and $GE = 2c\frac{c}{1-c} = \frac{2c^2}{1-c}$. Then, by Lemma 4, the segment GE is in the c-snowflake domain K. By construction,

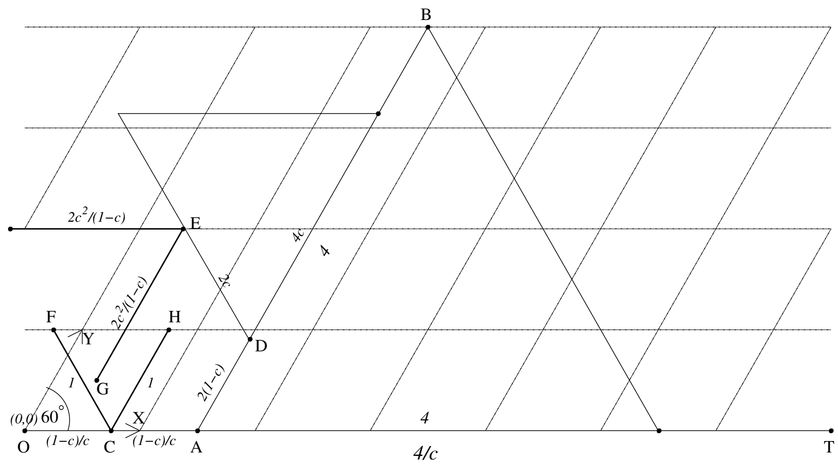

Figure 4:

$OA = \frac{4/c-4}{2} = \frac{2(1-c)}{c}$ and OA is a segment of an approximating polygon of K. Applying Lemma 4 for OA, we get that the segments CF and CH are in K, where C is the midpoint of OA and the angles OCF and HCA are $60°$ and $CF = CH = \frac{1-c}{c}\frac{c}{1-c} = 1$.

Using the lengths and directions of the above mentioned segments (and the obvious nice properties of the coordinate system with degree $60°$), we can easily calculate the coordinates of some vertices:

$$
\begin{aligned}
C &= \left(\frac{1-c}{c},\ 0\right), \\
D &= \left(\frac{2(1-c)}{c},\ 2(1-c)\right), \\
E &= \left(\frac{2(1-c)}{c} - 2c,\ 2(1-c)+2c\right) = \left(\frac{2(1-c-c^2)}{c},\ 2\right), \\
F &= \left(\frac{1-c}{c} - 1,\ 1\right) = \left(\frac{1-2c}{c},\ 1\right), \\
G &= \left(\frac{2(1-c-c^2)}{c},\ 2 - \frac{2c^2}{1-c}\right) = \left(\frac{2(1-c-c^2)}{c},\ \frac{2(1-c-c^2)}{1-c}\right), \\
H &= \left(\frac{1-c}{c},\ 1\right).
\end{aligned}
$$

Lemma 5 *If $\frac{1}{2} \le c \le \frac{\sqrt{5}-1}{2}$ then G is on the segment OH.*

Proof. The coordinates of G are $2(1-c-c^2)/(1-c)$ times the coordinates of H (in our coordinate system), so it is enough to check that $0 \le 2(1-c-c^2)/(1-c) \le$

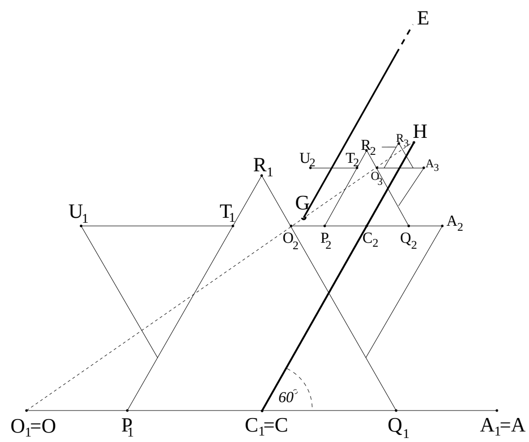

Figure 5:

1, which is clear since $c \leq \frac{\sqrt{5}-1}{2}$ implies that $1 - c - c^2 \geq 0$ and $c \geq \frac{1}{2}$ implies that $2(1 - c - c^2) \leq 1 - c$. $\square$

Proposition 6 *If $c \geq 1/\sqrt{2}$ then the segments CH and DE intersect each other, so the c-snowflake domain K is self-intersecting.*

Proof. We have $E^x \leq C^x = H^x < D^x$ $(2(1-c-c^2)/c \leq (1-c)/c < 2(1-c)/c$ even for $c \geq \frac{1}{2})$ hence the line CH separates E and D. On the other hand, we have $C^x+C^y < D^x+D^y = E^x+E^y \leq H^x+H^y$ $((1-c)/c < 2(1-c)/c+2(1-c) \leq (1-c)/c+1$ if $c \geq 1/\sqrt{2})$, which means that the line DE separates C and H. Therefore $CH \cap DE \neq \emptyset$ indeed. $\square$

Proposition 7 *If $\frac{\sqrt{5}-1}{2} \leq c \leq 1/\sqrt{2}$ then $CF \cap EG \neq \emptyset$, so K is self-intersecting.*

Proof. We have $F^x \leq G^x = E^x \leq C^x$ (since $(1-2c)/c \leq 2(1-c-c^2)/c$ for $c \leq 1/\sqrt{2}$ and $2(1-c-c^2)/c \leq (1-c)/c$ for $c \geq \frac{1}{2} \geq \frac{\sqrt{5}-1}{2}$), hence the line EG separates C and F. On the other hand, $G^x + G^y \leq 0$ (since $1 - c - c^2 \leq 0$ if $c \geq \frac{\sqrt{5}-1}{2}$) and $0 \leq C^x + C^y = F^x + F^y \leq E^x + E^y$ (by definition for any $0 < c < 1$), hence the line CF separates E and G. Therefore $CF \cap EG \neq \emptyset$.

Proposition 8 *If $\frac{1}{2} \leq c \leq \frac{\sqrt{5}-1}{2}$ then K is self-intersecting.*

Proof. Consider the part of K that was built on OA (see Fig. 4 and Fig. 5). Let $OP_1R_1Q_1A$ be the first generation descendant of OA (see Fig. 5). Let U_1T_1 be the upper side of the triangle we put on P_1R_1. We also use the notation A_1, C_1, O_1 for A, C and O.

Using a similarity from H with ratio c each point of Fig. 5 goes to the point denoted by the same letter and with index greater by 1. Note that, by self-similarity, everything we get is part of K. By Lemma 5, G is on the segment $OH = O_1O_2 \cup O_2O_3 \cup O_3O_4 \cup \ldots \cup \{H\}$. If G coincides with H then we are done (on the margin, this happens if and only if $c = \frac{1}{2}$). Otherwise G is on the segment O_nO_{n+1} for an $n \in \{1, 2, \ldots\}$.

We shall show that K is self-intersecting by proving that for this fixed n we have $U_nO_{n+1} \cap EG \neq \emptyset$.

Since $T_n^x = P_n^x$ and the condition $c \geq \frac{1}{2}$ implies that the segment U_nT_n is at least as long as O_nP_n, we have $U_n^x \leq O_n^x$. Clearly we also have $O_n^x \leq G^x = E^x \leq O_{n+1}^x$, thus the line EG separates U_n and O_{n+1}. On the other hand, clearly we have $G^y < O_{n+1}^y = U_n^y < H^y = 1 < 2 = E^y$, thus U_nO_{n+1} separates E and G. Therefore we have $U_nO_{n+1} \cap EG \neq \emptyset$, which completes the proof. □

Propositions 2, 6, 7 and 8 completes the proof of Theorem 1. □

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