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Difference functions of periodic measurable functions

PhD dissertation

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1996

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0. Introduction

In this dissertation we investigate problems of the following type:

Let f be a "nice" function. For which sets H is it true that

(∗) *if the difference functions $\Delta_h f(x) = f(x+h) - f(x)$ are "even nicer" for every $h \in H$ then f itself must be "even nicer"?*

0.1 Historical background

At the end of the 40's Mary L. Boas and R. P. Boas Jr. proved in an unpublished manuscript that a function f with continuous difference functions $\Delta_h f$, for each h, is continuous itself, provided that f is bounded on a set of positive measure. That is, if "nice" means "bounded on a set of positive measure" and "even nicer" means "continuous", then $H = \mathbf{R}$ satisfies (∗) .

N. G. de Bruijn generalized this result in [dB1] proving the following conjecture of Erdős:

Theorem 0.1.1. *If $f : \mathbf{R} \to \mathbf{R}$ is such that, for each h, $\Delta_h f$ is a continuous function, then f can be written in the form $g + G$, where g is continuous and G is additive (that is, $G(x+y) = G(x) + G(y)$ for each x, y).*

(This result is indeed a generalization of the theorem of Boas and Boas since a theorem of Ostrowski (see [Kes, O]) states that if an additive function is bounded on a set of positive measure, then it is of the form $G(x) = cx$.)

N. G. de Bruijn introduced the following notion: a class of real function $\mathcal{F}$ is said to have the *difference property*, if any real function f such that, for each h, $\Delta_h f \in \mathcal{F}$, is of the form $f = g + G$, where $g \in \mathcal{F}$ and G is additive. He proved in [dB1] and [dB2] that besides the class of continuous functions, the classes of periodic continuous, differentiable, analytic, absolutely continuous and Riemann-integrable functions also have the difference property. M. Laczkovich proved in [L2] that the class of point-wise discontinuous functions and some related classes also have the difference property.

Since a measurable additive function is necessarily linear, the difference property of these classes implies that if "nice" function means measurable function and "even nicer" function means a function from a class having the difference property then $H = \mathbf{R}$ satisfies (∗) .

A recent result of similar type by Trofimchuk ([T]) states that if $\Delta_h f$ is bounded and uniformly continuous for every $h \in \mathbf{R}$ then f is also uniformly continuous.

All of the results above concerned the case of $H = \mathbf{R}$. As far as I know, the first and only result answering a more general problem is due to M. Balcerzak, Z. Buczolich and M. Laczkovich. They proved in [BBL] the following:

Theorem 0.1.2. *For any subset H of the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, the following statements are equivalent:*

(i) *If $f : \mathbf{T} \to \mathbf{R}$ is continuous and $\Delta_h f$ is Lipschitz for every $h \in H$ then f is Lipschitz.*
(ii) *There is no proper F_σ subgroup of $\mathbf{T}$ containing H.*

Therefore, if "nice function" means continuous functions on $\mathbf{T}$ and "even nicer" means the Lipschitz property (with exponent 1) then the sets satisfying $(*)$ are those subsets H of $\mathbf{T}$ that cannot be covered by a proper F_σ subgroup of $\mathbf{T}$. We will generalize this result in Section 7.

The function $f(x) = x^2$ shows that the result of Trofimchuk does not remain true without the condition that $\Delta_h f$ is bounded, furthermore Theorem 0.1.2 is not true for functions on $\mathbf{R}$. However, it is also proved in [BBL] that the same result is true for functions on $\mathbf{R}$ if we require (as in the theorem of Trofimchuk) the boundedness of $\Delta_h f$ for any $h \in H$. To avoid these inconveniences we will usually work with functions defined on $\mathbf{T}$.

0.2. Notation

$\mathbf{R}, \mathbf{Z}$ and $\mathbf{N}$ will denote the sets of real numbers, integers and positive integers, respectively.

$\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the circle group. Sometimes we identify $\mathbf{T}$ with $[0, 1)$, sometimes we consider functions on $\mathbf{T}$ as periodic functions on $\mathbf{R}$ periodic modulo 1. If $x \in \mathbf{T}$ then by $|x|$ we mean $\min(x, 1 - x)$.

Let $\mathbf{G}$ be any of the groups $\mathbf{R}$ or $\mathbf{T}$. If $A, B \subset \mathbf{G}$ than we denote $A + B = \{a + b : a \in A, b \in B\}$. The sets $A - B$ and $-A$ are defined similarly. If $k \in \mathbf{N}$, the k-fold sum $A + \ldots + A$ is denoted by kA. For $H \subset \mathbf{G}$, the closure and the Lebesgue outer measure of H are denoted by $\mathrm{cl}(H)$ and $|H|$, respectively. By the measure of a set we mean its outer Lebesgue measure.

Δ_h denotes the difference operator with step h; that is, $\Delta_h f(x) = f(x + h) - f(x)$.

$\mathcal{C}$ is the class of continuous, $\mathcal{B}$ is the class of Borel and L_0 is the class of Lebesgue measurable functions on $\mathbf{T}$.

For $0 < p < \infty$, L_p denotes the class of those measurable functions f on $\mathbf{T}$ for which $\|f\|_p = \left(\int_T |f|^p\right)^{1/p} < \infty$.

L_∞ consists of the essentially bounded measurable functions on $\mathbf{T}$; that is, the measurable functions on $\mathbf{T}$ for which $\|f\|_\infty = \inf\{c : |f| \le c \text{ a. e.}\} < \infty$.

Lip^α denotes the class of functions f on $\mathbf{T}$ for which there exists an $L > 0$ such that $|f(x) - f(y)| \le L|x - y|^\alpha$ for every $x, y \in \mathbf{T}$. By a Lipschitz function we mean Lip^1 functions.

We denote by ACF the class of continuous functions with absolute convergent Fourier series on $\mathbf{T}$.

If $\mathcal{F}$ is a class of functions we denote by $\mathcal{F}^*$ the class of those functions that are equal to a function in $\mathcal{F}$ almost everywhere. If the elements of $\mathcal{F}$ are called P functions, where P is an arbitrary property (e.g. P=continuous) then we call the functions in $\mathcal{F}^*$ *essentially* P functions.

0.3. The notion $\mathcal{H}(\mathcal{F},\mathcal{G})$

Let $\mathcal{F}$ and $\mathcal{G}$ be classes of functions on $\mathbf{G}$ with $\mathcal{F}\supset\mathcal{G}$. We denote by $\mathcal{H}^0(\mathcal{F},\mathcal{G})$ the class of those subsets H of $\mathbf{G}$, for which there exists $f\in\mathcal{F}\setminus\mathcal{G}$ such that $\Delta_h f\in\mathcal{G}$ if and only if $h\in H$. That is,

$$\mathcal{H}^0(\mathcal{F},\mathcal{G})=\Big\{\{h\in\mathbf{G}:\Delta_h f\in\mathcal{G}\}:f\in\mathcal{F}\setminus\mathcal{G}\Big\}.$$

We denote by $\mathcal{H}(\mathcal{F},\mathcal{G})$ the class of sets that can be covered by a set in $\mathcal{H}^0(\mathcal{F},\mathcal{G})$. Note that $\mathcal{H}(\mathcal{F},\mathcal{G})$ is the class of sets $H\subset\mathbf{G}$ for which there exists $f\in\mathcal{F}\setminus\mathcal{G}$ such that $\Delta_h f\in\mathcal{G}$ for any $h\in H$. That is,

$$\mathcal{H}(\mathcal{F},\mathcal{G})=\Big\{H\subset\mathbf{G}:\exists f\in\mathcal{F}\setminus\mathcal{G}\ \ \Delta_h f\in\mathcal{G}\ \ \forall h\in H\Big\}.$$

Thus the family of sets satisfying $(*)$ is precisely the complement of $\mathcal{H}(\mathcal{F},\mathcal{G})$. For example, in our terminology, Theorem 0.1.2 means that

$$\mathcal{H}(\mathcal{C},\mathrm{Lip}^1)=\{\text{sets that can be covered by a proper } F_\sigma \text{ subgroup of } T\}.$$

Our main purpose is to determine $\mathcal{H}(\mathcal{F},\mathcal{G})$ for some pairs of classes of functions on $\mathbf{T}$ (e.g. L_0, $\mathcal{C}$, L_p, L_∞, Lip^α, ACF, $\mathcal{C}^*$, $(\mathrm{Lip}^\alpha)^*$, ACF^*, Baire-n functions). Since $\mathcal{H}^0(\mathcal{F},\mathcal{G})$ gives more information then $\mathcal{H}(\mathcal{F},\mathcal{G})$ we try to determine $\mathcal{H}^0(\mathcal{F},\mathcal{G})$ if it is possible. In most cases it is very difficult to determine even $\mathcal{H}(\mathcal{F},\mathcal{G})$. However, estimations can be also useful. For example, if $\mathcal{H}(\mathcal{F},\mathcal{G})\subset\mathcal{H}$, then for proving that a function $f\in\mathcal{F}$ is in $\mathcal{G}$ it is enough to check that $\Delta_h f$ is in $\mathcal{G}$ for a set of h's that is not in $\mathcal{H}$.

This notion is closely related to the notion of difference property. For example, as the next lemma shows, for periodic continuous functions our notion is a kind of generalization of the difference property.

Lemma 0.3.1. *If $\mathcal{G}\subset\mathcal{C}$ and $\mathcal{G}$ is invariant under the addition with constants then the following statements are equivalent:*

(i) $\mathbf{T}\notin\mathcal{H}(\mathcal{C},\mathcal{G})$,

(ii) *$\mathcal{G}$ has the difference property.*

Proof.

(i)$\Rightarrow$(ii): Suppose that $\Delta_h f\in\mathcal{G}$ for any h. Then, since $\mathcal{G}\subset\mathcal{C}$ and $\mathcal{C}$ has the difference property, f can be written in the form $g+G$, where $g\in\mathcal{C}$ and G is additive. Thus, for any h, $\Delta_h f=\Delta_h g+C$, where C is a constant. Hence $\Delta_h g=\Delta_h f-C\in\mathcal{G}$ for any h, which implies - using $\mathbf{T}\notin\mathcal{H}(\mathcal{C},\mathcal{G})$ - that $g\in\mathcal{G}$.

(ii)$\Rightarrow$(i): Assume that $f\in\mathcal{C}\setminus\mathcal{G}$ and $\Delta_h f\in\mathcal{G}$ for every $h\in\mathbf{T}$. Then the difference property of $\mathcal{G}$ implies that $f=g+G$ where $g\in\mathcal{G}$ and G is additive. Since $G=f-g$ is continuous, periodic and additive, $G=0$. Thus $f\in\mathcal{G}$, which is a contradiction. ☺

The following easy facts will be used frequently.

Lemma 0.3.2. *If $\mathcal{F} \supset \mathcal{G}$ and $\mathcal{G}$ is a translation invariant group of functions on $\mathbf{T}$ (with pointwise addition), then each element of $\mathcal{H}^0(\mathcal{F}, \mathcal{G})$ is a subgroup of $\mathbf{T}$.*

Proof. By definition

$$\Delta_{-h} f(x) = f(x-h) - f(x) = -\Delta_h f(x-h),$$

thus if $\Delta_h f \in \mathcal{G}$ then also $\Delta_{-h} f \in \mathcal{G}$. In addition

$$\Delta_{h_1+h_2} f(x) = f(x+h_1+h_2) - f(x+h_1) + f(x+h_1) - f(x) = \Delta_{h_2} f(x+h_1) + \Delta_{h_1} f(x),$$

therefore if $\Delta_{h_1} f, \Delta_{h_2} f \in \mathcal{G}$ then also $\Delta_{h_1+h_2} f \in \mathcal{G}$. ☺

Monotonicity Lemma 0.3.3. *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{G}$ then*

$$\mathcal{H}^0(\mathcal{F}_1, \mathcal{G}) \supset \mathcal{H}^0(\mathcal{F}_2, \mathcal{G}) \quad \text{and} \quad \mathcal{H}(\mathcal{F}_1, \mathcal{G}) \supset \mathcal{H}(\mathcal{F}_2, \mathcal{G}). \text{ ☺}$$

Lemma 0.3.4 (Triangle inequality). *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ then*

$$\mathcal{H}(\mathcal{F}_1, \mathcal{F}_3) \subset \mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) \ \cup \ \mathcal{H}(\mathcal{F}_2, \mathcal{F}_3).$$

Proof. Suppose that $H \in \mathcal{H}(\mathcal{F}_1, \mathcal{F}_3)$ but $H \notin \mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ and $H \notin \mathcal{H}(\mathcal{F}_2, \mathcal{F}_3)$. Then $H \in \mathcal{H}(\mathcal{F}_1, \mathcal{F}_3)$ implies that there exists a function $f \in \mathcal{F}_1 \setminus \mathcal{F}_3$ such that $\Delta_h f \in \mathcal{F}_3$ for any $h \in H$. Since $H \notin \mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ and $\mathcal{F}_3 \subset \mathcal{F}_2$, f cannot be in $f \in \mathcal{F}_1 \setminus \mathcal{F}_2$, therefore $f \in \mathcal{F}_2 \setminus \mathcal{F}_3$, which contradicts $H \notin \mathcal{H}(\mathcal{F}_2, \mathcal{F}_3)$. ☺

Lemma 0.3.5. *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ and $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) \subset \mathcal{H}(\mathcal{F}_2, \mathcal{F}_3)$ then*

$$\mathcal{H}(\mathcal{F}_1, \mathcal{F}_3) = \mathcal{H}(\mathcal{F}_2, \mathcal{F}_3).$$

Proof. This is trivial from Lemma 0.3.3 and Lemma 0.3.4. ☺

0.4. Thin sets in harmonic analysis

The classes $\mathcal{H}(\mathcal{F}, \mathcal{G})$ are often related to some classes of thin sets in harmonic analysis. Now we define those classes that will arise in our results. Detailed explanation of this topic can be read in the monographs [B], [Z] and [KS], in the recent research papers [HMP] and [Ka] or in the recent topical survey [BKR].

A set $H \subset \mathbf{T}$ is called a *Dirichlet set* if there exists an increasing sequence of integers (q_n) such that $|\sin q_n \pi x| \to 0$ uniformly on H; that is, there exists a sequence (ε_n) converging to zero such that for any $x \in H$

$$|\sin q_n \pi x| < \varepsilon_n \qquad (\forall\, n \in \mathbf{N}\ ).$$

A set $H \subset \mathbf{T}$ is called a *pseudo-Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $n_0(x)$ such that

$$|\sin q_n \pi x| < \varepsilon_n \qquad \text{if} \quad n \geq n_0(x).$$

A set $H \subset T$ is called an *N-set* if there exists a trigonometric series that is absolutely convergent on H but is not absolutely convergent everywhere; that is, if there exist sequences (a_n) and (b_n) such that $\sum_{n=1}^{\infty}(|a_n| + |b_n|) = \infty$ but for any $x \in H$,

$$\sum_{n=1}^{\infty} \big(\ |a_n \cos(2\pi n x)| + |b_n \sin(2\pi n x)|\ \big) < \infty.$$

The family of Dirichlet sets, pseudo-Dirichlet sets and N-sets are denoted by $\mathcal{D}$, $p\mathcal{D}$ and $\mathcal{N}$, respectively.

We denote by $\mathcal{F}_\sigma$ the family of those sets that can be covered by a proper F_σ subgroup of $\mathbf{T}$. Therefore with our notation Theorem 0.1.2 means that $\mathcal{H}(\mathcal{C}, \mathrm{Lip}^1) = \mathcal{F}_\sigma$.

Lemma 0.4.1.

$$\mathcal{D}\ \subset\ p\mathcal{D}\ \subset\ \mathcal{N}\ \subset\ \mathcal{F}_\sigma.$$

Proof.

$\mathcal{D} \subset p\mathcal{D}$: This is trivial from the definition.

$p\mathcal{D} \subset \mathcal{N}$: Let $H \in p\mathcal{D}$. Then there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $n_0(x)$ such that $|\sin q_n \pi x| < \varepsilon_n$ if $n \geq n_0(x)$. Then we can clearly choose a subsequence ε_{n_k} such that $\sum_k \varepsilon_{n_k} < \infty$. Then $\sum_{k=1}^{\infty} \sin(2\pi q_{n_k} x)$ is a trigonometric series that is absolutely convergent on H but is not absolutely convergent everywhere.

$\mathcal{N} \subset \mathcal{F}_\sigma$: Suppose that the Fourier series $\sum_{n=1}^{\infty} A_n(x)$ is absolutely convergent on a set H (where $A_n(x) = a_n \cos(2\pi n x) + b_n \sin(2\pi n x)$) but is not absolutely convergent everywhere. Let H_{km} be the set of those points where $\sum_{n=1}^{k} |A_n(x)| \leq m$. Then H_{km} is clearly closed for any $k, m \in \mathbf{N}$ and putting $B = \cup_{m=1}^{\infty} \cap_{k=1}^{\infty} H_{km}$, B is the set of those points where $\sum A_n$ is absolutely convergent. It is easy to check that the set of these points is a subgroup of $\mathbf{T}$. Therefore B is a proper F_σ subgroup of $\mathbf{T}$. On the other hand $H \subset B$, so the proof is complete. ☺

Remark 0.4.2. All the inclusions in Lemma 0.4.1 are proper. For the (not too difficult examples) for $\mathcal{D} \neq p\mathcal{D}$ and $p\mathcal{D} \neq \mathcal{N}$ see e.g. [Ka]. It is much more difficult to construct a set from $\mathcal{F}_\sigma \setminus \mathcal{N}$. M. Laczkovich and I. Ruzsa constructed such a set in [LRu].

1. Continuous difference functions of Borel functions

In this chapter we investigate the set of h's for which the difference functions $\Delta_h f$ of a function f from a given Borel class are continuous. (That is, we consider $\mathcal{H}^0(\mathcal{F},\mathcal{C})$ where $\mathcal{F}$ is a Borel class). We will see that the characterization must be difficult: these sets need not be Borel sets even if f is Baire 2. As we shall see, in general we have more information on the class $\mathcal{H}(\mathcal{F},\mathcal{G})$ than on $\mathcal{H}^0(\mathcal{F},\mathcal{G})$. For this reason, in the sequel we will not make special efforts to determine $\mathcal{H}^0(\mathcal{F},\mathcal{G})$.

Notation 1.1. For an $f:\mathbf{G}\to\mathbf{R}$ function, where $\mathbf{G}=\mathbf{R}$ or $\mathbf{T}$, we denote by H_f the set of h's for which $\Delta_h f$ is continuous.

Proposition 1.2. *If $f:\mathbf{R}\to\mathbf{R}$ is a Borel function then H_f is a coanalytic set.*

Proof. By definition

$$\mathbf{R}\setminus H_f=\big\{h:\exists x\in\mathbf{R}\ \exists k\in\mathbf{N}\ \forall n\in\mathbf{N}\ \exists|\delta|<1/n: \\ \big|(f(x+\delta+h)-f(x+\delta))-(f(x+h)-f(x))\big|\ge 1/k\big\}.$$

Since f is a Borel function, the set

$$A_{n,k}=\big\{(h,x,\delta)\in\mathbf{R}^3:\ |\delta|<1/n,\ \big|(f(x+\delta+h)-f(x+\delta))-(f(x+h)-f(x))\big|\ge 1/k\big\}$$

is a Borel set. Hence

$$\mathbf{R}\setminus H_f=\operatorname{proj}_x\big(\bigcup_{k\in\mathbf{N}}\bigcap_{n\in\mathbf{N}}\operatorname{proj}_\delta(A_{n,k})\big)$$

is an analytic set. Therefore H_f is a coanalytic set. ☺

Proposition 1.3. *If an $f:\mathbf{R}\to\mathbf{R}$ function has a point of continuity but is not continuous everywhere, then the set H_f is a discrete (additive) subgroup of* $\mathbf{R}$.

Proof. The set H_f is clearly an (additive) subgroup of $\mathbf{R}$. Assume that it is not discrete. Then it is dense.

Let $\omega(x)$ be the oscillation of f at x; that is, $\omega=\overline{f}-\underline{f}$. The function $\omega(x)$ is upper semi-continuous, since it is a sum of two upper semi-continuous functions, so the sets of the form $\{x:\omega(x)\ge c\}$ are closed for any $c\in\mathbf{R}$. On the other hand $\omega(x)$ is periodic modulo h for any $h\in H_f$, since $f(x+h)=\Delta_h f(x)+f(x)$, and $\Delta_h f$ is continuous everywhere.

Therefore for any $c\in\mathbf{R}$ the set $\{x:\omega(x)\ge c\}$ is closed and is periodic modulo a dense set, so these sets must be either empty or the whole real line, which implies that $\omega(x)$ is constant. Since f has a point of continuity, this constant must be 0. Then f is continuous, contrary to our hypothesis. ☺

Proposition 1.4. *The sets of form H_f where $f : \mathbf{R} \to \mathbf{R}$ is a Baire 1 function but is not continuous everywhere, are the discrete (additive) subgroups of* $\mathbf{R}$.

Proof. Since every Baire 1 function has a point of continuity, using the previous proposition, we get that any such H_f is a discrete subgroup of $\mathbf{R}$. On the other hand, the characteristic functions of the discrete subgroups of $\mathbf{R}$ show that all of these sets are sets of type H_f for suitable Baire 1 but not continuous functions. ☺

Corollary 1.5. *The sets of form H_f where $f : \mathbf{T} \to \mathbf{R}$ is a Baire 1 function but is not continuous everywhere, are the discrete (additive) subgroups of* $\mathbf{T}$; *that is,*

$$\mathcal{H}^0(\mathit{Baire}\ 1,\ \mathcal{C}) = \big\{\{k/n : k = 0, 1, \ldots, n-1\} : n \in \mathbf{N}\big\},$$

$$\mathcal{H}(\mathit{Baire}\ 1,\ \mathcal{C}) = \{\mathbf{T} \text{ and the finite subsets of } \mathbf{T} \cap \mathbf{Q}\}. ☺$$

Theorem 1.6. *There exists an F_σ-set $E \subset \mathbf{R}$ such that the set of periods of E is not Borel.*

Proof. Let P and Q be disjoint perfect subsets of $\mathbf{R}$ such that $0 \notin P$ and the points of $P \cup Q$ are linearly independent over $\mathbf{Q}$. It is well-known that such sets exist and also that there exists a G_δ-set $A \subset P \times Q$ such that its projection onto the first coordinate axis is not a Borel-set. Let $B = (P \times Q) \setminus A$. Then B is clearly an F_σ-set. Let

$$E = \big\{x_1 + \ldots + x_n + y : n \in \mathbf{N},\ y \in Q,\ x_i \in P,\ (x_i, y) \in B\ (i = 1, \ldots, n)\big\} \cup Q.$$

First we prove that E is an F_σ-set. Let

$$F_n = \big\{(x_1, y, \ldots, x_n, y) \in \mathbf{R}^{2n} : (x_i, y) \in B\ (i = 1, \ldots, n)\big\}.$$

Since B is F_σ it is easy to check that so is F_n. On the other hand

$$E = \Big(\bigcup_{n \in \mathbf{N}} h_n(F_n)\Big) \cup Q$$

where $h_n(x_1, y_1, \ldots, x_n, y_n) = x_1 + \ldots + x_n + y_1$. Since the sets F_n are F_σ-sets, the functions h_n are continuous and Q is closed, we get that E is F_σ.

Now we prove that the set of periods of E is non-Borel. For this we prove the following:

For every $u \in P$, E is periodic modulo $u \Leftrightarrow (u, y) \in B$ for all $y \in Q$

$\Leftarrow$: Assume that $u \in P$ and $(u, y) \in B$ for all $y \in Q$. If $a \in E$ then either $a = x_1 + \ldots + x_n + y$ where $n \in \mathbf{N},\ y \in Q,\ x_i \in P,$ and $(x_i, y) \in B$, or $a \in Q$. In the first case $a + u = x_1 + \ldots + x_n + u + y \in E$, since $u \in P$ and $(u, y) \in B$. In the second case also $a + u = u + a \in E$, since $a \in Q$, $u \in P$ and $(u, a) \in B$.

$\Rightarrow$: Assume that E is periodic modulo $u \in P$ and let $y \in Q$ arbitrary. We prove that $(u, y) \in B$. Since $y \in Q \subset E$ and E is periodic modulo u we get $u + y \in E$. So either $u + y$ can be written as $u + y = x_1 + \ldots + x_n + y'$ where $n \in \mathbf{N},\ y' \in Q,\ x_i \in P,$ and $(x_i, y') \in B$ or $u + y \in Q$. Since the points of $P \cup Q$ are linearly independent over

Q and $0 \notin P$ the second case cannot happen and in the first case $u+y = x_1+\ldots+x_n+y'$ implies that $n = 1, u = x_1$ and $y = y'$. Therefore $(u, y) = (x_1, y') \in B$.

Denote by M the set of periods of E. According to our last observation

$$M \cap P = \{u \in P : \forall y \in Q \ (u, y) \in B\}.$$

That is,

$$(\mathbf{R} \setminus M) \cap P = \{u \in P : \exists y \in Q \ (u, y) \in A\},$$

which is the projection of A onto the first coordinate axis. So according to the choice of A, the set $(\mathbf{R} \setminus M) \cap P$ is not Borel. Since P is Borel, this implies that M is not Borel. ☺

Corollary 1.7. *There exists a Baire 2 function f (either on $\mathbf{R}$ or $\mathbf{T}$) such that H_f is not a Borel set.*

Proof. Let f be the characteristic function of the set E, constructed in the previous theorem. Since E is an F_σ-set, f is a Baire 2 function. On the other hand it is easy to see that $\Delta_h f$ is continuous if and only if E is periodic modulo h. Therefore H_f is not a Borel set. Since f is periodic we can consider f as a function on $\mathbf{T}$. ☺

Remark 1.8. Corollary 1.7 shows that the characterization of the sets of $\mathcal{H}^0(\mathcal{B}, \mathcal{C})$ (where $\mathcal{B}$ is the class of Borel, $\mathcal{C}$ is the class of continuous functions on $\mathbf{T}$) must be difficult. However, for the sets of $\mathcal{H}(\mathcal{B}, \mathcal{C})$ we have a characterization. As we shall prove later (Theorem 6.5) $H \in \mathcal{H}(\mathcal{B}, \mathcal{C})$ if and only if H can be covered by the set of periods of a nonempty proper Borel subset of $\mathbf{T}$.

2. Changes on null-sets. Essentially continuous functions

Notation 2.1. We recall that if $\mathcal{F}$ is a class of functions we denote by $\mathcal{F}^*$ the class of those functions that are equal to a function in $\mathcal{F}$ almost everywhere.

If the elements of $\mathcal{F}$ are called P functions, where P is an arbitrary property (e.g. P=continuous) then we will call the functions in $\mathcal{F}^*$ *essentially* P functions.

In this section we investigate what happens if we replace a class of functions $\mathcal{F}$ by $\mathcal{F}^*$. We will see that in the most important cases the corresponding class $\mathcal{H}$ either remains the same or becomes much more interesting.

First we state a simple lemma.

Lemma 2.2. *If $\mathcal{G} \subset \mathcal{C}$ then $\mathcal{G}^* \cap \mathcal{C} = \mathcal{G}$.*

Proof. If $f \in \mathcal{G}^* \cap \mathcal{C}$ then there exists a function $\tilde{f} \in \mathcal{G}$ that is almost everywhere equal to f. But they are both continuous so this implies that $f = \tilde{f}$, that is $f \in \mathcal{G}$. ☺

Proposition 2.3. *If $\mathcal{C} \supset \mathcal{F} \supset \mathcal{G}$ then*

$$\mathcal{H}(\mathcal{F}, \mathcal{G}) = \mathcal{H}(\mathcal{F}, \mathcal{G}^*) = \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*).$$

Proof.

$\underline{\mathcal{H}(\mathcal{F}, \mathcal{G}) \subset \mathcal{H}(\mathcal{F}, \mathcal{G}^*)}$: If $H \in \mathcal{H}(\mathcal{F}, \mathcal{G})$ then there exists a function $f \in \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}$ for any $h \in H$. Applying Lemma 2.2, we get that $f \notin \mathcal{G}^*$. Therefore $f \in \mathcal{F} \setminus \mathcal{G}^*$ and $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$, which shows that $H \in \mathcal{H}(\mathcal{F}, \mathcal{G}^*)$.

$\underline{\mathcal{H}(\mathcal{F}, \mathcal{G}) \supset \mathcal{H}(\mathcal{F}, \mathcal{G}^*)}$: If $H \in \mathcal{H}(\mathcal{F}, \mathcal{G}^*)$ then there exists a function $f \in \mathcal{F} \setminus \mathcal{G}^* \subset \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$. Since $f \in \mathcal{F} \subset \mathcal{C}$ we get $\Delta_h f \in \mathcal{C}$. Applying Lemma 2.2, we get that $\Delta_h f \in \mathcal{G}$. Therefore $f \in \mathcal{F} \setminus \mathcal{G}$ and $\Delta_h f \in \mathcal{G}$ for any $h \in H$, which shows that $H \in \mathcal{H}(\mathcal{F}, \mathcal{G})$.

$\underline{\mathcal{H}(\mathcal{F}, \mathcal{G}^*) \subset \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)}$: This follows from the monotonicity-lemma.

$\underline{\mathcal{H}(\mathcal{F}, \mathcal{G}^*) \supset \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)}$: If $H \in \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$ then there exists a function $f \in \mathcal{F}^* \setminus \mathcal{G}^*$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$. Since $f \in \mathcal{F}^*$ there exists a function $\tilde{f} \in \mathcal{F}$ such that $f = \tilde{f}$ a.e. Since $f \notin \mathcal{G}^*$ we get $\tilde{f} \notin \mathcal{G}^*$, hence $\tilde{f} \in \mathcal{F} \setminus \mathcal{G}^*$. On the other hand $\Delta_h f \in \mathcal{G}^*$ implies that $\Delta_h \tilde{f} \in \mathcal{G}^*$. Therefore $H \in \mathcal{H}(\mathcal{F}, \mathcal{G}^*)$. ☺

Proposition 2.4. *If $\mathcal{G} \subset \mathcal{F} \subset L_0$, $\mathcal{G} \subset \mathcal{C}$ and $\mathcal{G}$ contains the constant 0 function, then*

$$\mathcal{H}^0(\mathcal{F}^*, \mathcal{G}) \supset \{\text{additive subgroups of zero measure}\}.$$

Proof. Let A be an additive subgroup with zero measure. Let f be its characteristic function.

Since $f = 0$ a.e. and $0 \in \mathcal{G} \subset \mathcal{F}$ we get $f \in \mathcal{F}^*$. If $a \in A$ then $\Delta_a f = 0 \in \mathcal{G}$. If $a \notin A$ then $\Delta_a f$ is a non-constant function with finite range, so it cannot be continuous, hence it is not in $\mathcal{G}$. Therefore f shows that $A \in \mathcal{H}^0(\mathcal{F}^*, \mathcal{G})$. ☺

Remark 2.5. Later we will show (Theorem 6.3) that if $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$ then equality holds in Proposition 2.4.

In the sequel we will work with classes of functions of the following two types:

(i) classes of measurable functions that are invariant for changes on zero-sets (that is, $\mathcal{F} = \mathcal{F}^*$);

(ii) classes of continuous functions that contain the constant 0 function.

Instead of $\mathcal{H}(\mathcal{F}, \mathcal{G})$ we will usually investigate $\mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$. If $\mathcal{F}$ and $\mathcal{G}$ are both of type (i), then these classes of sets are trivially the same, if $\mathcal{F}$ and $\mathcal{G}$ are both of type (ii) then Proposition 2.3 shows that these classes of sets are equal.

If $\mathcal{F}$ is of type (i) and $\mathcal{G}$ is of type (ii) then these classes are usually not equal (we will show that $\mathcal{H}(\mathcal{F}, \mathcal{G})$ contains $\mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$), but in this case, as Proposition 2.4 shows, $\mathcal{H}(\mathcal{F}, \mathcal{G})$ is "too big". In this case it is much more interesting to investigate $\mathcal{H}(\mathcal{F}, \mathcal{G}^*)$, which is the same as $\mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$.

Lemma 2.6. *Let A be an additive subgroup of $\mathbf{R}$ and let S be a dense union of translated copies of A. Suppose that we have a function $h : \mathbf{R} \to \mathbf{R}$ and continuous functions $l_a : \mathbf{R} \to \mathbf{R}$ for all $a \in A$ such that $\Delta_a h|S = l_a|S$ for any $a \in A$.*

Then there exists a function $\tilde{h} : \mathbf{R} \to \mathbf{R}$ such that $\tilde{h}|S = h|S$ and $\Delta_a \tilde{h} = l_a$ for every $a \in A$.

Moreover, if h is bounded, then we can choose $\tilde{h}$ to be also bounded.

Proof. Choose a point from those translated copies of A that are not in S. Denote this set by B. Then every $x \notin S$ can be uniquely written in the form $x = \beta + b$ where $\beta \in B$ and $b \in A$. For such an $x \notin S$ let $\tilde{h}(x) = l_b(\beta)$. For $x \in S$ let $\tilde{h}(x) = h(x)$.

Now we need to prove that $\Delta_a \tilde{h}(x) = l_a(x)$ for all $a \in A$ and $x \in \mathbf{R}$. We know that this equality holds if $x \in S$. If $x \notin S$ then $x = \beta + b$ ($\beta \in B,\ b \in A$), so using the definition of $\tilde{h}$:

$$\Delta_a \tilde{h}(x) = \Delta_a \tilde{h}(\beta + b) = \tilde{h}(\beta + b + a) - \tilde{h}(\beta + b) = l_{b+a}(\beta) - l_b(\beta).$$

So it is enough to prove that

$$l_a(\beta + b) = l_{b+a}(\beta) - l_b(\beta) \tag{2}$$

for any $\beta \in B$ and $a, b \in A$.

If β is not in B but in S then (2) holds. Indeed, in this case $\beta + b$ is also in S so $l_a(\beta + b) = \Delta_a h(\beta + b)$, $l_{b+a}(\beta) = \Delta_{b+a} h(\beta)$ and $l_b(\beta) = \Delta_b h(\beta)$ and a trivial calculation shows that $\Delta_a h(\beta + b) = \Delta_{b+a} h(\beta) - \Delta_b h(\beta)$.

Therefore (2) holds on a dense subset of $\mathbf{R}$. On the other hand both sides of this equation are continuous, so this implies that (2) holds on the whole real line, thus specially it also holds on B, which completes the proof of the first part of this lemma.

Now we only need to prove that if h is bounded, then $\tilde{h}$ is also bounded. Suppose that $|h| \le M$. Then, since $\Delta_a h|S = l_a|S$, we get $|l_a| \le 2M$ on S for any $a \in A$. But l_a is continuous, S is dense, so this implies that $|l_a| \le 2M$ everywhere. Thus, according to the definition of $\tilde{h}$, we get $|\tilde{h}| \le 2M$. ☺

Main Lemma 2.7. *Suppose that $H \subset \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ is a measurable function and $\Delta_h f$ is essentially continuous for any $h \in H$.*

Then there exists a function $\tilde{f}$ such that $\tilde{f} = f$ a.e. and $\Delta_h f$ is continuous for any $h \in H$.

Moreover, if f is bounded, then we can choose $\tilde{f}$ to be also bounded.

Proof. Let A be the additive subgroup of $\mathbf{R}$ generated by H. Then clearly $\Delta_a f$ is essentially continuous also for any $a \in A$. Thus for each $a \in A$ there exists a continuous function l_a such that $\Delta_a f = l_a$ a.e. Then

$$f(x+a) = f(x) + l_a(x) \quad \text{a.e. (for any fixed } a \in A). \tag{3}$$

Let

$$S = \{x : f \text{ has a finite approximative limit at } x\}.$$

Since f is a measurable function the set S has full measure.

For any $x \in S$, the right-hand-side of (3) has a finite approximative limit at x, so the left-hand-side, which is a.e. equal to it, also has a finite approximative limit at x. That is, if $x \in S$ and $a \in A$ then $x + a \in S$. Therefore S is a dense (since it has full measure) union of translated copies of A.

Let

$$f_1(x) = \begin{cases} \mathrm{limappr}_x f & \text{if } x \in S \\ f(x) & \text{if } x \notin S. \end{cases}$$

If f is bounded then so is f_1. Since f is measurable it is almost everywhere approximately continuous, so $f_1 = f$ a.e. This implies that their approximative limits are equal everywhere. Thus for any $x \in S$ we get $f_1(x) = \mathrm{limappr}_x f = \mathrm{limappr}_x f_1$, which implies that f_1 (and thus also $\Delta_a f_1$) is approximately continuous at the points of S. On the other hand $\Delta_a f_1 = \Delta_a f$ a.e. and $\Delta_a f = l_a$ a.e, so $\Delta_a f = l_a$ a.e.

Hence for any $x \in S$ and $a \in A$ we get

$$\Delta_a f_1(x) = \mathrm{limappr}_x \Delta_a f_1(x) = \mathrm{limappr}_x l_a = l_a(x).$$

Now applying the previous lemma, changing f_1 on the complement of S, we can get a function $\tilde{f}$ such that $\Delta_a \tilde{f} = l_a$ for any $a \in A$. Thus $\tilde{f} = f$ a.e. (since $\tilde{f} = f_1$ on S, S has full measure and $f_1 = f$ a.e.) and $\Delta_h \tilde{f}$ is continuous for any $h \in H \subset A$. Moreover $\tilde{f}$ is bounded, if f is bounded. ☺

Remark 2.8. Let $\mathcal{H}$ be any class of sets, $\mathcal{F}$ be a class of functions of type (i) (see Remark 2.5) and $\mathcal{G}$ be a class of functions of type (ii). Then if we want to prove that $\mathcal{H}(\mathcal{F}, \mathcal{G}^*) \subset \mathcal{H}$, then according to the definition, we have to prove that if $f \in \mathcal{F} \setminus \mathcal{G}^*$ and

$\Delta_h f \in \mathcal{G}^*$ for any $h \in H$, then $H \in \mathcal{H}$. Applying the Main Lemma we can use not only that $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$, but also the stronger property that $\Delta_h f \in \mathcal{G}$.

Corollary 2.9. *If $\mathcal{G} \subset \mathcal{F} \subset L_0$ and $\mathcal{G} \subset \mathcal{C}$, then*

$$\mathcal{H}(\mathcal{F}^*, \mathcal{G}^*) \subset \mathcal{H}(\mathcal{F}^*, \mathcal{G}). \square$$

Theorem 2.10. *If $f : \mathbf{R} \to \mathbf{R}$ is measurable, $\Delta_h f$ is essentially continuous for any $h \in \mathbf{R}$ then f is also essentially continuous.*

Proof. According to the Main Lemma, there exists a function $\tilde{f}$ such that $\tilde{f} = f$ a.e. and $\Delta_h \tilde{f}$ is continuous for any $h \in \mathbf{R}$. Then applying the Theorem of de Bruijn (Theorem 0.1.1) we get that $\tilde{f}$ is a sum of a continuous and an additive functions. But since $\tilde{f}$ is measurable, this implies that $\tilde{f}$ is continuous, so f is essentially continuous. $\square$

Remark 2.11. For periodic functions this theorem is the first step for a stronger result. We will prove (Theorem 5.6) that if f is a measurable function on $\mathbf{T}$ and $\Delta_h f$ is essentially continuous for any $h \in H$, and H cannot be covered by a proper F_σ-subgroup of $\mathbf{T}$, then f is essentially continuous.

At this point one can hope that the class of essentially continuous functions has the difference property; that is, for any $f : \mathbf{R} \to \mathbf{R}$, if $\Delta_h f \in \mathcal{C}^*$ for any $h \in \mathbf{R}$ then f is a sum of an essentially continuous and an additive functions. However, this is not the case. More precisely the following is true:

Theorem 2.12. *Supposing the continuum hypothesis, the class of essentially continuous functions does not have the difference property.*

Proof. Assuming CH Sierpiński ([S]) constructed a non-measurable function $S : \mathbf{R} \to \{0, 1\}$ such that for any fixed $h \in \mathbf{R}$, $\Delta_h S(x) = 0$ with exception of at most a countable number of x-values.

Then clearly $\Delta_h S \in \mathcal{C}^*$ for any $h \in S$. But if S was the sum of an essentially continuous and an additive functions, then the additive function would be essentially bounded, which would mean that it is constant. Then S would be essentially continuous but S is not measurable. $\square$

However, the class $\mathcal{C}^*$ has a weaker property. We say that a class $\mathcal{F}$ has the *weak difference property* if every function $f : \mathbf{G} \to \mathbf{R}$ for which $\Delta_h f \in \mathcal{F}$ for every $h \in \mathbf{G}$ admits a decomposition $f = g + H + S$ with $g \in \mathcal{F}$, H additive, and S satisfying the condition that for every $h \in \mathbf{G}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbf{G}$.

Lemma 2.13. *Suppose that (i) $\mathcal{F} \supset \mathcal{G}$ are classes of measurable functions on $\mathbf{G}$ (where $\mathbf{G} = \mathbf{T}$ or $\mathbf{R}$), (ii) $\mathcal{G}$ is a group that contains the linear functions, and (iii) $\mathcal{F}^*$ has the weak difference property.*

Then $\mathcal{G}^*$ has the weak difference property if and only if $\mathbf{G} \notin \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$.

Proof. Assume that $\mathcal{G}^*$ has the weak difference property but $\mathbf{G} \in \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$. Then there exists a function $f \in \mathcal{F}^* \setminus \mathcal{G}^*$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in \mathbf{G}$. Since $\mathcal{G}^*$ has the weak difference property, this implies that $f = g + H + S$ where $g \in \mathcal{G}^*$, H is additive, and S satisfies the condition that for every $h \in \mathbf{T}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbf{G}$.

Let $l = f - g = H + S$. Then l is measurable and $\Delta_h l$ is constant a.e. for any $h \in \mathbf{G}$. Thus, by the Main Lemma (2.7), there exists a function $\tilde{l}$ such that $\tilde{l} = l$ a.e. and $\Delta_h \tilde{l}$ is constant everywhere. Then $\tilde{l} - \tilde{l}(0)$ is a measurable additive function, so $\tilde{l}$ is linear, thus $\tilde{l} \in \mathcal{G}$. Since $f = g + \tilde{l}$ a.e., $g \in \mathcal{G}^*$ and $\mathcal{G}$ is a group, this implies that $f \in \mathcal{G}^*$, which is a contradiction.

Now we prove that if $\mathbf{G} \notin \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$ then $\mathcal{G}^*$ has the weak difference property. Suppose that for the function $f : \mathbf{G} \to \mathbf{R}$, $\Delta_h f \in \mathcal{G}^*$ for every $h \in \mathbf{G}$. Then, since $\mathcal{G}^* \subset \mathcal{F}^*$, f has a decomposition $f = g + H + S$ with $g \in \mathcal{F}^*$, H additive, and S satisfying the condition that for every $h \in \mathbf{G}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbf{G}$. Then $\Delta_h f = \Delta_h g + \Delta_h H + \Delta_h S$. Since $\Delta_h f \in \mathcal{G}^*$, $\Delta_h S = 0$ a.e. and $\Delta_h H$ is constant, this implies that also $\Delta_h g \in \mathcal{G}^*$ for any $h \in \mathbf{G}$. Therefore, since $\mathbf{G} \notin \mathcal{H}(\mathcal{F}^*, \mathcal{G}^*)$, $g \in \mathcal{G}^*$. ☺

Theorem 2.14. *The class of essentially continuous functions has the weak difference property.*

Proof. In [L1] M. Laczkovich proved that the class of measurable functions has the weak difference property. Then, by Lemma 2.13, Theorem 2.10 implies that $\mathcal{C}^*$ also has the weak difference property. ☺

Proposition 2.15. *If $f : \mathbf{R} \to \mathbf{R}$ is measurable but not essentially continuous and $\Delta_h f$ is continuous for a dense set of h-s, then*

$$\limsup_x f = +\infty \quad \text{and} \quad \liminf_x f = -\infty \qquad (\forall x \in \mathbf{R}).$$

Proof. We prove the $\limsup f = +\infty$ part of the statement, the proof of the other statement is the same.

Let

$$H_f = \{h : \Delta_h f \text{ is continuous}\}.$$

Since $f(x+h) = \Delta_h f(x) + f(x)$ and $\Delta_h f$ is continuous for $h \in H_f$ it follows that $\overline{f} - f$ is periodic modulo each $h \in H_f$. Thus if $\overline{f}(x_0) = +\infty$ for any $x_0 \in \mathbf{R}$ then $\limsup_x f = +\infty$ on a dense set, which implies that $\limsup_x f = +\infty$ everywhere. Therefore we can assume that $\overline{f}$ is finite everywhere.

For a fixed $h \in H_f$, the function $\overline{f} - f$ is periodic modulo h, so $\overline{f}(x+h) - f(x+h) = \overline{f}(x) - f(x)$, which implies that $\Delta_h \overline{f} = \Delta_h f$. Therefore for any $h \in H_f$, $\Delta_h \overline{f}$ is also continuous. Thus $H_{\overline{f}}$ is also dense. On the other hand $\overline{f}$ is upper semi-continuous, so it is Baire-1, so it has a point of continuity. Then according to Proposition 1.3, $\overline{f}$ is continuous.

Since $\overline{f} - f$ is measurable and its periods form a dense set, $\overline{f} - f$ is constant a.e. Thus, since $\overline{f}$ is continuous, f is essentially continuous, contradicting our assumption. ☺

Theorem 2.16. *If f is measurable and essentially bounded, and if $\Delta_h f$ is essentially continuous for a dense set of h-s, then f is essentially continuous.*

Proof. Let
$$H = \{h : \Delta_h f \in \mathcal{C}^*\}.$$
Since f is essentially bounded there exists an f_1 such that $f_1 = f$ a.e. and f_1 is bounded. Then for any $h \in H$, $\Delta_h f_1$ is also essentially continuous. Applying the Main Lemma we can take a bounded function $\tilde{f}$ such that $\tilde{f} = f_1$ a.e. and $\Delta_h \tilde{f}$ is continuous for any $h \in H$. Since H is dense, the last proposition shows that this can happen only if $\tilde{f}$ is essentially continuous. But then f, which is almost everywhere equal to $\tilde{f}$, is also essentially continuous. ☺

Corollary 2.17.

$$\mathcal{H}^0(L_\infty, \mathcal{C}^*) = \{\text{finite subgroups of } \mathbf{T}\},$$

$$\mathcal{H}(L_\infty, \mathcal{C}^*) = \{\text{finite subsets of } \mathbf{T} \cap \mathbf{Q}\}.$$

Proof. Since the subsets of $\mathbf{T}$ that can be covered by a finite subgroup of $\mathbf{T}$ are the finite subsets of $\mathbf{T} \cap \mathbf{Q}$ it is enough to prove the first equality.

$\subset$: This is an immediate consequence of the previous theorem.

$\supset$: Let G be a finite subgroup of $\mathbf{T}$. Then it is easy to see that G is of the form $G = \{0, 1/n, 2/n, \ldots, (n-1)/n\}$. Let $f(x) = \operatorname{sgn}(\sin(2\pi n x))$. Then clearly $f \in L_\infty \setminus \mathcal{C}^*$ and $\{h : \Delta_h f \in \mathcal{C}^*\} = G$. ☺

3. Not essentially bounded periodic measurable functions with many continuous difference functions.

$$\mathcal{H}(L_p, ACF^*) = \mathcal{N}$$

In this chapter we prove that for any $p \geq 1$, $\mathcal{H}(L_p, ACF^*) = \mathcal{N}$.

Lemma 3.1. *If $d_1 \geq d_2 \geq \ldots \geq 0$ and $\sum d_n = \infty$, then $\sum \min(d_n, 1/n) = \infty$.*

Proof. We can assume that $d_n > 1/n$ for infinitely many n, since otherwise $\min(d_n, 1/n) = d_n$ for n large enough.

Then we can choose a subsequence d_{n_k} such that $n_k \geq 2n_{k-1}$ and $d_{n_k} > 1/n_k$ for every k. Then

$$\sum \min(d_n, 1/n) = \sum_k \sum_{m=n_{k-1}+1}^{n_k} \min(d_m, 1/m) \geq \sum_k \sum_{m=n_{k-1}+1}^{n_k} 1/n_k = \sum_k \frac{n_k - n_{k-1}}{n_k} \geq$$
$$\geq \sum_k \frac{1}{2} = \infty. \quad ☺$$

Lemma 3.2. *If $a_n \geq 0$ $(n = 1, 2, \ldots)$ and $\sum a_n = \infty$, then there exists a sequence (b_n) such that*

(i) $$0 \leq b_n \leq a_n \qquad (n = 1, 2, \ldots),$$

(ii) $$\sum b_n = \infty,$$

(iii) $$\sum b_n^q < \infty \qquad \text{for any } q > 1, \text{ and}$$

(iv) $$N \sum_{n=N+1}^{\infty} b_n/n \leq 4 \qquad (N = 1, 2, \ldots).$$

Proof. If $a_n \to 0$ then we can rearrange (a_n) such that $a_{\phi(1)} \geq a_{\phi(2)} \geq \ldots$ where ϕ is a permutation of $\mathbf{N}$. In this case let $c_{\phi(n)} = \min(a_{\phi(n)}, 1/n)$. Then, applying the previous lemma for $d_n = a_{\phi(n)}$, we get $\sum c_n = \sum c_{\phi(n)} = \infty$. On the other hand

$\sum c_n^q = \sum c_{\phi(n)}^q < \infty$ for any $q > 1$, since $c_{\phi(n)} \le 1/n$. Furthermore clearly $0 \le c_n \le a_n$ $(n = 1, 2, \ldots)$.

If $a_n \not\to 0$ then there exists an $\varepsilon > 0$ and a subsequence a_{n_m} such that $a_{n_m} > \varepsilon$. Then let $c_{n_m} = \varepsilon/m$ and let the other terms of the sequence (c_n) be 0. Then in this case clearly also $0 \le c_n \le a_n$ $(n = 1, 2, \ldots)$, $\sum c_n = \infty$ and $\sum c_n^q < \infty$ for any $q > 1$.

Let $s_k = \sum_{j=2^k+1}^{2^{k+1}} c_j$. If $2^k < n \le 2^{k+1}$ then let

$$b_n = \begin{cases} c_n & \text{if } s_k < 1 \\ c_n/s_k & \text{if } s_k \ge 1. \end{cases}$$

Then clearly $0 \le b_n \le c_n \le a_n$ for any n, so (i) and (iii) hold.

If $s_k \ge 1$ for a k then by the definition of b_n

$$\sum_{n=2^k+1}^{2^{k+1}} b_n = \sum_{n=2^k+1}^{2^{k+1}} \frac{c_n}{s_k} = \sum_{n=2^k+1}^{2^{k+1}} \frac{c_n}{\sum_{j=2^k+1}^{2^{k+1}} c_j} = 1.$$

So if $s_k \ge 1$ for infinitely many k then $\sum b_n$ is divergent. On the other hand if $s_k \ge 1$ only for finitely many k then $b_n = c_n$ for n large enough, so in this case the divergence of $\sum c_n$ implies the divergence of $\sum b_n$. Therefore (ii) also holds.

Finally we need to prove (iv). If $2^K \le N < 2^{K+1}$ then

$$N \sum_{n=N+1}^{\infty} b_n/n < 2^{K+1} \sum_{n=2^K+1}^{\infty} b_n/n = 2^{K+1} \sum_{k=K}^{\infty} \sum_{j=2^k+1}^{2^{k+1}} b_j/j <$$

$$< 2^{K+1} \sum_{k=K}^{\infty} \frac{\sum_{j=2^k+1}^{2^{k+1}} b_j}{2^k} \le 2^{K+1} \sum_{k=K}^{\infty} \frac{1}{2^k} = 4. \quad ☺$$

Lemma 3.3. *If the nonnegative sequence (b_n) satisfies the conditions (ii) and (iv) then no function $f \in L_\infty$ can have the Fourier series*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \cos(2\pi n x). \tag{4}$$

Proof. For getting a contradiction assume that the function $f \in L_\infty$ has the Fourier series (4). For a fixed N we use the following notation:

$$x_N = \frac{1}{6N}, \quad S_N = \sum_{n=1}^{N} b_n, \quad m_N(x) = f(x) - \sum_{n=1}^{N} b_n \cos(2\pi n x) \sim \sum_{n=N+1}^{\infty} b_n \cos(2\pi n x).$$

Then for any $x \in [0, x_N]$

$$f(x) = \Big(\sum_{n=1}^{N} b_n \cos(2\pi n x)\Big) + m_N(x) \ge \frac{1}{2} S_N + m_N(x).$$

On the other hand, using the term by term integrability of Fourier series,

$$\left|\int_0^{x_N} m_N(x)dx\right| = \left|\sum_{n=N+1}^{\infty}\int_0^{x_N} b_n\cos(2\pi nx)\right| = \left|\sum_{n=N+1}^{\infty} b_n\frac{\sin(2\pi nx_N)}{2\pi n}\right| \le \sum_{n=N+1}^{\infty}\frac{b_n}{6n}.$$

Therefore, using condition (iv), the absolute value of the average of $m_N(x)$ on the interval $[0, x_N]$ is

$$\left|\frac{\int_0^{x_N} m_N(x)dx}{x_N}\right| \le N\sum_{n=N+1}^{\infty}\frac{b_n}{n} \le 4.$$

Thus on a subset H_N of $[0, x_N]$ with positive Lebesgue measure $m_N(x) \ge -4$, which implies that

$$f(x) \ge \frac{1}{2}S_N + m_N(x) \ge \frac{1}{2}S_N - 4 \qquad (x \in H_N).$$

On the other hand, according to (ii), $S_N \to \infty$, which means that $f \notin L_\infty$. ☺

Theorem 3.4. *For any N-set $H \subset \mathbf{R}$ there exists a modulo 1 periodic function $f : \mathbf{R} \to \mathbf{R}$ such that $f \in L_p$ for every finite p but $f \notin L_\infty$, and $\Delta_h f$ is continuous and has an absolutely convergent Fourier series for any $h \in H$.*

Proof. It is known (see e.g. [Z] Vol. I, p. 236) that if H is an N-set, then it is also an N_s-set; that is, there exists a nonnegative divergent series $\sum a_n$ such that

(5) $$\sum a_n|\sin\pi nh| < \infty \qquad (\forall h \in H).$$

Applying Lemma 3.2, we can get a sequence (b_n) satisfying the conditions (i), (ii), (iii) and (iv). Using condition (i), (5) implies that

(6) $$\sum b_n|\sin\pi nh| < \infty \qquad (\forall h \in H).$$

Let A denote the set of all h-s for which (6) holds. It is easy to see that A is an additive subgroup of $\mathbf{R}$ and $H \subset A$. Let $\tilde{f}$ be a modulo 1 periodic complex valued function with the Fourier series

$$\tilde{f}(x) \sim \sum_{n=1}^{\infty} b_n e^{2\pi inx}.$$

According to the Riesz-Fischer theorem, condition (iii) implies that such a function exists in L_2. Moreover this condition implies that this function is in L_p for any $p > 0$ (see e.g. [Z], proof of the Hausdorff-Young theorem, Vol. II, p. 101-103). Let $\overline{f} = \mathrm{Re}\,\tilde{f}$. Then clearly also $\overline{f} \in \cap_{p>0} L_p$. On the other hand $\overline{f}(x) \sim \sum_{n=1}^{\infty} b_n\cos(2\pi nx)$, so according to Lemma 3.3, $\overline{f} \notin L_\infty$.

For a fixed $h \in A$ the Fourier series of $\tilde{f}(x+h)$ is

$$\tilde{f}(x+h) \sim \sum_{n=1}^{\infty}(b_n e^{2\pi inh})e^{2\pi inx},$$

so

$$\Delta_h \tilde{f}(x) \sim \sum_{n=1}^{\infty} b_n (e^{2\pi i n h} - 1) e^{2\pi i n x}. \tag{7}$$

On the other hand

$$\left| b_n (e^{2\pi i n h} - 1) e^{2\pi i n x} \right| = 2 b_n |\sin \pi n h|.$$

Thus (6) implies that the right-hand side of (7) is uniformly convergent, so denoting it by $\tilde{l}_h(x)$, the function $\tilde{l}_h(x)$ is continuous on $\mathbf{R}$.

Let S be the set of points where the averages of the partial sums of the Fourier series (the Fejér means) of $\tilde{f}$ converge to $\tilde{f}(x)$. According to Lebesgue's theorem S contains the Lebesgue points of $\tilde{f}$, so its complement is a null-set. Changing $\tilde{f}$ on this null-set we can make $\tilde{f}(x)$ to be equal to the limit of the Fejér means at each points where it exists, so we can assume that S is also the set of points where the Fejér means converge.

Since the Fejér means of $\tilde{l}_h$ converge to $\tilde{l}_h(x)$ everywhere, the Fejér means of $\tilde{f}(x)$ and $\tilde{f}(x+h)$ converge simultaneously, thus $x \in S$ if and only if $x + h \in S$. Therefore S is a dense union of translated copies of A. If $x \in S$ then, according to (7), $\Delta_h \tilde{f}(x)$ and $\tilde{l}_h(x)$ are the limits of the averages of the partial sums of the same Fourier series, thus $\Delta_h \tilde{f}(x) = \tilde{l}_h(x)$ if $x \in S$. Therefore denoting the real part of $\tilde{l}_h$ by l_h we get

$$\Delta_h \overline{f}(x) = l_h(x) \qquad (x \in S,\ h \in A).$$

Now applying Lemma 2.6, there exists a function $f(x)$ on $\mathbf{R}$ such that $f|S = \overline{f}|S$ and

$$\Delta_h f(x) = l_h(x) \qquad (x \in \mathbf{R},\ h \in A).$$

In particular $\Delta_1 f(x) = l_1(x) = 0$, which implies that f is periodic modulo 1; for any $h \in H \subset A$ we get $\Delta_h f = l_h$, which shows that $\Delta_h f$ is continuous for any $h \in H$. Since f and $\overline{f}$ are equal almost everywhere and $\overline{f} \in (\cap_{p>0} L_p) \setminus L_\infty$ we get $f \in (\cap_{p>0} L_p) \setminus L_\infty$. ☺

Notation 3.5. We recall that we denote by ACF the class of continuous functions with absolute convergent Fourier series on $\mathbf{T}$.

Note that ACF^* consists of the (not necessary) continuous functions with absolutely convergent Fourier series on $\mathbf{T}$.

We use the notation $\mathcal{N}$ for the class of N-subsets of $\mathbf{T}$.

Corollary 3.6. *If $ACF \subset \mathcal{F} \subset L_\infty$ and $0 < p < \infty$ then*

$$\mathcal{H}(L_p, \mathcal{F}^*) \supset \mathcal{N}.$$

Theorem 3.7.

$$\mathcal{H}(L_1, ACF^*) \subset \mathcal{N}.$$

Proof. Let $H \in \mathcal{H}(L_1, ACF^*)$. Then there exists a function $f \in L_1 \setminus ACF^*$ such that $\Delta_h f \in ACF^*$ for any $h \in H$. Let the Fourier series of f be

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \qquad (c_{-k} = \overline{c}_k).$$

Since the Fourier series of f is not absolutely convergent we get $\sum_{k=1}^{\infty} |c_k| = \infty$. It is easy to see that the Fourier series of $\Delta_h f$ is

$$\Delta_h f \sim \sum_{k=-\infty}^{\infty} c_k (e^{2\pi ikh} - 1) e^{2\pi ikx}.$$

Let

$$E = \Big\{ h \in \mathbf{T} \ : \ \sum_{k=1}^{\infty} |c_k| |e^{2\pi ikh} - 1| < \infty \Big\}.$$

Then, since $\Delta_h f$ has an absolutely convergent Fourier series for any $h \in H$, we get $H \subset E$.

In [HMP] B. Host, J.-F. Méla and F. Parreau call a set of type

$$\Big\{ h \in \mathbf{T} \ : \ \sum_{j=0}^{\infty} a_j |e^{2\pi i n_j h} - 1| < \infty \Big\} \tag{8}$$

an H_1 group if n_j is a sequence of positive integers and $a_j \geq 0$ (p. 44, 2.3.1). They proved that if $\sum_{j=0}^{\infty} a_j = \infty$, then the H_1 group defined by (8) is a proper subgroup of $\mathbf{T}$. They also proved that, for a Borel subset of $\mathbf{T}$, it is equivalent to be an N-set and to be contained in an H_1 proper subgroup.

Using these facts and notation, E is an H_1 proper subgroup of $\mathbf{T}$, thus (since E is clearly an F_σ set so it is also Borel) E is an N-set. Since $H \subset E$ we get that H is also an N-set. ☺

Corollary 3.8. *For any $p \geq 1$*

$$\mathcal{H}(L_p, ACF^*) = \mathcal{N}.$$

Proof. This is trivial from Corollary 3.6, Theorem 3.7 and from the Monotonicity Lemma. ☺

Corollary 3.9. *ACF has the difference property.*

Proof. By Proposition 2.3, the Monotonicity Lemma and Theorem 3.7,

$$\mathcal{H}(\mathcal{C}, ACF) = \mathcal{H}(\mathcal{C}^*, ACF^*) \subset \mathcal{H}(L_1, ACF^*) \subset \mathcal{N}.$$

Hence $\mathbf{T} \notin \mathcal{H}(\mathcal{C}, ACF)$, so according to Lemma 0.3.1, ACF has the difference property. ☺

4. An application: Measurable decompositions of measurable functions

Let $f : \mathbf{R} \to \mathbf{R}$ be a real function and let $\alpha_1, \dots, \alpha_n$ be given real numbers. We say that $f = f_1 + \dots + f_n$ is an $(\alpha_1, \dots, \alpha_n)$*-decomposition* of f if f_i is periodic modulo α_i for every $i = 1, \dots, n$. We say that $f = f_1 + \dots + f_n + p$ is an $(\alpha_1, \dots, \alpha_n)$*-quasi-decomposition* of f if p is a polynomial of degree $< n$ and f_i is periodic modulo α_i for every $i = 1, \dots, n$. (The decomposition or the quasi-decomposition is said to be continuous or measurable if all the functions f_i are continuous or measurable resp.) If f has an $(\alpha_1, \dots, \alpha_n)$-decomposition or an $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition, then it is easy to see that $\Delta_{\alpha_1} \dots \Delta_{\alpha_n} f = 0$. A class F of real functions is said to have the *decomposition property* if, for every $f \in \mathcal{F}$ and $\alpha_1, \dots, \alpha_n \in \mathbf{R}$, the equation $\Delta_{\alpha_1} \dots \Delta_{\alpha_n} f = 0$ implies that f has an $(\alpha_1, \dots, \alpha_n)$-decomposition with functions in F.

These notions were introduced in [LR1] by M. Laczkovich and Sz. Révész. They proved in [LR1] and [LR2] that a series of important classes of real functions have the decomposition property (e.g. $L_\infty(\mathbf{R})$, the class of bounded Lipschitz functions and the class of bounded continuous functions). On the other hand it is easy to see that the class of all $\mathbf{R} \to \mathbf{R}$ functions and the class of all continuous functions do not have the decomposition property. ($f(x) = x$ is a counter-example for both classes.) It was asked in [LR2] whether it is true that if f is continuous and has a measurable $(\alpha_1, \dots, \alpha_n)$-decomposition then f also has a continuous $(\alpha_1, \dots, \alpha_n)$-decomposition or at least a continuous $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition.

If f is uniformly continuous, then it is an easy consequence of a result in [LR2], that the answer is affirmative. However, using Theorem 3.4, we shall prove that in the general case the answer is negative. We will construct two periodic measurable functions with continuous and unbounded sum.

We will use the following known (and easy to prove) fact ([LR2]):

Proposition 4.0. *No non-constant polynomial (more generally no function f with $\lim_{x\to\infty} |f| = \infty$) can be the sum of finitely many periodic measurable functions.*

Proposition 4.1. *Let $\alpha_1, \dots, \alpha_n$ be real numbers such that α_i/α_j is irrational for every $1 \le i < j \le n$. Then a function $f : \mathbf{R} \to \mathbf{R}$ can have at most one measurable $(\alpha_1, \dots, \alpha_n)$-decomposition, apart from additive constants and changes on a null-set.*

Proof. Clearly it is enough to prove that if

$$(9) \qquad f_1 + \dots + f_n = 0 \qquad \text{a.e.,}$$

where f_j is measurable and periodic modulo α_j for all j, then f_n is constant a.e. We prove it by induction. For $n = 1$ it is obvious. Let $n \geq 2$. Since f_1 is periodic modulo α_1 the equation (9) implies

$$\Delta_{\alpha_1} f_2 + \ldots + \Delta_{\alpha_1} f_n = 0 \qquad \text{a.e.}$$

Now using the induction assumption we get that $\Delta_{\alpha_1} f_n = c$ a.e. Then the function $g(x) = f_n(x) - (c/\alpha_1)x$ is measurable and periodic modulo α_1 except a set of measure zero. Thus $(c/\alpha_1)x$ is the sum of two measurable periodic functions, which implies, using Proposition 4.0, that $c = 0$. Therefore the measurable function f_n has two incommensurable periods (except a set of measure zero), which implies that f_n is constant a.e. ☺

Proposition 4.2. *If $f = f_1 + \ldots + f_n$, $f \in L_\infty$, the functions f_i are measurable and periodic modulo α_i ($i = 1, \ldots, n$), where α_i/α_j is irrational for every $i \neq j$, then the functions $f_1, \ldots, f_n$ are also in L_∞.*

Proof. It is proved in [LR1] that the class of functions L_∞ has the decomposition property. By Proposition 4.1, this implies the statement. ☺

Proposition 4.3. *If f has a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition and a measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition, then f has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition, too.*

Proof. Suppose that $f_1 + \ldots + f_n = f = g_1 + \ldots + g_n + p$, where the functions f_i are measurable, the functions g_i are continuous and p is a polynomial. Then $p = (f_1 - g_1) + \ldots + (f_n - g_n)$, so using Proposition 4.0, p must be constant, so $g_1 + \ldots + g_{n-1} + (g_n + p)$ is a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of f. ☺

Theorem 4.4. *If a uniformly continuous function f has a measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition, then it also has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition.*

Proof. It is proved in [LR2] (4.2.Thm.) that a function f has a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition with linear p if and only if $\Delta_{\alpha_1} \ldots \Delta_{\alpha_n} f = 0$ and f is uniformly continuous. Thus Proposition 4.3 implies this theorem.

Remark 4.5. By Proposition 4.1, this theorem implies that if the sum of finitely many measurable periodic functions with pairwise incommensurable periods is uniformly continuous, then these functions must be essentially continuous.

Remark 4.6. These results do not remain true if we omit the word 'uniformly'. This is the main result of this section. For bounded continuous functions, however, these results become true again, since the class of bounded continuous functions has the decomposition property.

Theorem 4.7. *For any N-set $H \subset \mathbf{R}$ there exists a modulo 1 periodic function $h \in (\cap_{p>0} L_p) \setminus L_\infty$ and there are modulo α periodic functions $g_\alpha \in (\cap_{p>0} L_p)$ for all $\alpha \in H$ such that $g_\alpha + h$ is continuous for all $\alpha \in H$.*

Proof. Let h be the function constructed in Theorem 3.4. Since $\Delta_\alpha h$ is continuous ($h \in H$) there exists a continuous function f_α for each $\alpha \in H$ such that $\Delta_\alpha f_\alpha = \Delta_\alpha h$. Then the functions $g_\alpha = f_\alpha - h$ ($\alpha \in H$) are periodic modulo α, are in $(\cap_{p>0} L_p)$ and the functions $g_\alpha + h = f_\alpha$ are continuous. ☺

Theorem 4.8. *The sum of two periodic measurable functions can be continuous and unbounded.*

Proof. Using Theorem 4.7 for $H = \{\alpha\}$, where α is irrational, we get measurable functions h, g periodic modulo 1 and α resp. such that $h \notin L_\infty$ and $f = g + h$ is continuous. But according to Proposition 4.2, f cannot be bounded. ☺

Corollary 4.9. *There exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ that has a measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition but does not have a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition, nor has a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition.*

Proof. The unbounded continuous sum of the two periodic measurable functions constructed above clearly has a measurable (α_1, α_2)-decomposition but does not have a continuous (α_1, α_2)-decomposition (since otherwise it would be bounded). On the other hand, by Proposition 4.3, this function cannot have an $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition either. ☺

Finally, using Theorem 2.15, we shall prove that in Theorem 4.8. neither functions can be essentially bounded.

Proposition 4.10. *If f and g are measurable periodic functions, f is essentially bounded and $f + g$ is essentially continuous, then g is also essentially bounded. Moreover, if the periods are incommensurable, then f and g are essentially continuous.*

Proof. We can suppose that the periods of f and g are 1 and β resp.

If β is rational, then $f + g$ is periodic. Then $f + g$ is essentially bounded, so g is also essentially bounded.

Now let β be irrational. We have $\Delta_\beta f = \Delta_\beta (f + g)$, thus $\Delta_\beta f$ is essentially continuous. The same is clearly also true for any $\alpha = n + k\beta$. Since the set of these α-s is dense, we can apply Theorem 2.15 for getting that f is essentially continuous. Then clearly g is also essentially continuous and essentially bounded. ☺

Corollary 4.11. *The sum of a periodic measurable and a periodic measurable essentially bounded function cannot be continuous and unbounded.* ☺

5. $\mathcal{H}(\mathcal{F},\mathcal{G}) \subset \mathcal{F}_\sigma$ for the classes $L_0, L_p, L_\infty, \mathcal{C}^*, ACF^*$ and $(\mathrm{Lip}^\alpha)^*$

Consider the following classes of functions

$$L_0 \supset L_p \supset L_\infty \supset \mathcal{C}^* \supset ACF^* \text{ and } (\mathrm{Lip}^\alpha)^*.$$

(If $\alpha > \frac{1}{2}$ then, by a theorem of S. Bernstein (see e.g. [Z] Vol. I, p. 240), we have also $ACF \supset \mathrm{Lip}^\alpha$.) In this section we prove that for any pair of these classes of functions we have

$$\mathcal{H}(\mathcal{F},\mathcal{G}) \subset \mathcal{F}_\sigma \qquad (\mathcal{F} \supset \mathcal{G}).$$

(We recall that $\mathcal{F}_\sigma$ is the class of subsets of $\mathbf{T}$ that can be covered by a proper F_σ subgroup of $\mathbf{T}$.) By the Monotonicity Lemma, it is enough to prove this for $\mathcal{F} = L_0$. If $\mathcal{G} \subset \mathcal{F} \subset \mathcal{C}$ then, by Proposition 2.3, everything remains the same without $*$; that is, we have the same results for $\mathcal{C}, ACF$ and Lip^α.

We will need the following well-known lemma:

Lemma 5.0. *If $f : \mathbf{T} \to \mathbf{R}$ is a measurable function and (a_n) is a sequence of reals converging to 0, then we can choose a subsequence (a_{n_k}) such that*

$$\lim_{k\to\infty} f(x + a_{n_k}) = f(x) \qquad \text{for a. e. } x \in \mathbf{T}.$$

Proposition 5.1. *The sets in $\mathcal{H}^0(L_0, L_p)$, for any $0 < p \le \infty$, are F_σ subgroups of $\mathbf{T}$.*

Proof. Since the classes of functions in this proposition are translation invariant groups, the group property of the sets is clear.

Thus it is enough to prove that for any $f \in L_0$ the set

$$H = \Big\{ h : \|\Delta_h f\|_p \le K \Big\}$$

is closed for any $0 < p \le \infty$.

Suppose that $h_n \in H$ and $h_n \to h$. By Lemma 5.0, we can choose a subsequence (h_{n_k}) such that $f(x + h_{n_k}) \to f(x + h)$ for a. e. $x \in T$. Then clearly also $\Delta_{h_{n_k}} f \to \Delta_h f$ a.e..

If $p = \infty$ then $h_{n_k} \in H$ means that $|\Delta_{h_{n_k}} f| \le K$ a.e., thus also $|\Delta_h f| \le K$ a.e. which means that $h \in H$. If $p < \infty$ then $h_{n_k} \in H$ means that $\int |\Delta_{h_{n_k}} f|^p \le K^p$, so by the Fatou lemma also $\int |\Delta_h f|^p \le K^p$ which means that $h \in H$. ☺

Proposition 5.2. *If $f : \mathbf{T} \to \mathbf{R}$ is measurable and $\Delta_h f$ is essentially bounded for each $h \in \mathbf{T}$ then f is also essentially bounded. (That is, $\mathbf{T} \notin \mathcal{H}(L_0, L_\infty)$.)*

Proof. Let

$$H_n = \Big\{ h : |\Delta_h f| \le n \ \text{ a. e.} \Big\}.$$

Since f is measurable, H_n is also measurable, so $\cup H_n = \mathbf{T}$ implies that there exists an n such that H_n has positive measure. Then by a theorem of Steinhaus the set $H_n + H_n$ contains a neighborhood of 0. Thus kH_n contains the whole $\mathbf{T}$ if k is big enough. Hence, for any $h \in \mathbf{T}$, $|\Delta_h f| \le kn$ a. e.

Therefore, denoting nk by K,

$$\Big\{ (x,h) : x, h \in T, |f(x+h) - f(x)| > K \Big\}$$

is a measurable subset of $\mathbf{T} \times \mathbf{T}$ and each of its horizontal section is a 0-set. Thus, by Fubini's theorem, almost each of its vertical section is also a 0-set, which means that for almost any $x \in T$, $|f(x+h) - f(x)| \le K$ for almost every h. Therefore, choosing a proper x_0, $|f(x)| \le |f(x_0)| + K$ for almost every x, which means that f is essentially bounded. ☺

Corollary 5.3. *The class $L_\infty(\mathbf{T})$ has the weak difference property.*

Proof. This is trivial from Lemma 2.13 and Proposition 5.2. ☺

Proposition 5.4. *If $0 < p < \infty$, $f : \mathbf{T} \to \mathbf{R}$ is measurable, and $\Delta_h f \in L_p$ for each $h \in \mathbf{T}$ then also $f \in L_p$. (That is, $\mathbf{T} \notin \mathcal{H}(L_0, L_p)$ for $0 < p < \infty$.)*

Proof. M. Laczkovich proved in [L1] that the class L_p has the weak difference property for any $0 < p < \infty$, which means that if $\Delta_h f \in L_p$ for each $h \in \mathbf{T}$ then $f = g + H + S$ where $g \in L_p$, H is additive and for any h we have $\Delta_h S = 0$ a.e. Thus $\Delta_h(f-g)$ is constant almost everywhere for any h, so it is essentially continuous for any h. Since $f - g$ is measurable, by Theorem 2.10, $f - g$ is also essentially continuous, which implies that $f = g + (f - g)$ is in L_p. ☺

From the last three propositions we get the following:

Theorem 5.5. *For any $0 < p \le \infty$,*

$$\mathcal{H}^0(L_0, L_p) \subset \Big\{ \text{the proper } F_\sigma \text{ subgroups of } \mathbf{T} \Big\},$$
$$\mathcal{H}(L_0, L_p) \subset \mathcal{F}_\sigma. \text{ ☺}$$

Now, applying the Triangle-inequality lemma (Lemma 0.3.4), we can prove easily the following two theorems combining Theorem 5.5 with results of the previous sections.

Theorem 5.6.

$$\mathcal{H}(L_0, \mathcal{C}^*) \subset \mathcal{F}_\sigma.$$

Proof. By the Triangle-inequality lemma

$$\mathcal{H}(L_0, \mathcal{C}^*) \subset \mathcal{H}(L_0, L_\infty) \cup \mathcal{H}(L_\infty, \mathcal{C}^*).$$

By Theorem 5.5, $\mathcal{H}(L_0, L_\infty) \subset \mathcal{F}_\sigma$, by Corollary 2.16, $\mathcal{H}(L_\infty, \mathcal{C}^*) = \{\text{finite subsets of } \mathbf{T} \cap \mathbf{Q}\} \subset \mathcal{F}_\sigma$, so we completed the proof. ☺

Theorem 5.7.

$$\mathcal{H}(L_0, ACF^*) \subset \mathcal{F}_\sigma.$$

Proof. By the Triangle-inequality lemma $\mathcal{H}(L_0, ACF^*) \subset \mathcal{H}(L_0, L_1) \cup \mathcal{H}(L_1, ACF^*)$. By Theorem 5.5, $\mathcal{H}(L_0, L_1) \subset \mathcal{F}_\sigma$; by Theorem 3.7, $\mathcal{H}(L_1, ACF^*) \subset \mathbf{N} \subset \mathcal{F}_\sigma$, so we completed the proof. ☺

Theorem 5.8. *If* $0 < \alpha \leq 1$ *then*

$$\mathcal{H}\big(L_0, (\mathrm{Lip}^\alpha)^*\big) \subset \mathcal{F}_\sigma.$$

Proof. M. Balcerzak, Z. Buczolich and M. Laczkovich proved in [BBL] that $\mathcal{H}(\mathcal{C}, \mathrm{Lip}^\alpha) \subset \mathcal{F}_\sigma$ (Theorem 1.4). (Actually, they stated it only for $\alpha = 1$ but their proof works without any modification for Lip^α functions as well.) Then by Proposition 2.3 also $\mathcal{H}\big(\mathcal{C}^*, (\mathrm{Lip}^\alpha)^*\big) \subset \mathcal{F}_\sigma$. Then using the Monotonicity Lemma, the Triangle-inequality lemma and Theorem 5.6 we get

$$\mathcal{H}\big(L_0, (\mathrm{Lip}^\alpha)^*\big) \subset \mathcal{H}(L_0, \mathcal{C}^*) \cup \mathcal{H}\big(\mathcal{C}^*, (\mathrm{Lip}^\alpha)^*\big) \subset \mathcal{F}_\sigma. \text{☺}$$

Now we can summarize our results:

Theorem 5.9. *If* $\mathcal{F} \supset \mathcal{G}$ *are any of the classes* L_0, L_p $(0 < p \leq \infty)$, $\mathcal{C}^*, ACF^*$ *or* $(\mathrm{Lip}^\alpha)^*$ $(0 < \alpha \leq 1)$ *then*

$$\mathcal{H}(\mathcal{F}, \mathcal{G}) \subset \mathcal{F}_\sigma.$$

If $\mathcal{F} \supset \mathcal{G}$ *are any of the classes* $\mathcal{C}$, ACF *or* Lip^α $(0 < \alpha \leq 1)$ *then also*

$$\mathcal{H}(\mathcal{F}, \mathcal{G}) \subset \mathcal{F}_\sigma.$$

Proof. This follows from Theorems 5.5-5.8 using the Monotonicity Lemma and Proposition 2.3. ☺

6. Applications of $\mathcal{H}(L_0, \mathcal{C}^*) \subset \mathcal{F}_\sigma$. Functions with continuous differences

In this section we determine $\mathcal{H}^0(L_0, C)$ and $\mathcal{H}(L_0, C)$ and prove some other results we announced earlier. All these results are strongly based on the fact that $\mathcal{H}(L_0, \mathcal{C}^*) \subset \mathcal{F}_\sigma$.

Theorem 6.1.

$$\mathcal{H}^0(L_0, C) = \Big\{ \text{ subgroups of } \mathbf{T} \text{ with 0 measure} \Big\},$$
$$\mathcal{H}(L_0, C) = \Big\{ \text{sets that can be covered by a subgroup of } \mathbf{T} \text{ with 0 measure} \Big\}.$$

Proof. Clearly it is enough to prove the upper equality. We have already proved the inclusion $\supset$ in Proposition 2.4 so now we only need to prove the other inclusion.

Let $H \in \mathcal{H}^0(L_0, \mathcal{C})$. It is clear that H is a subgroup of $\mathbf{T}$. For getting a contradiction, assume that H has positive measure. Since $H \in \mathcal{H}^0(L_0, \mathcal{C})$, there exists an $f \in L_0 \setminus \mathcal{C}$ such that $\Delta_h f$ is continuous for any $h \in H$.

Since H has positive measure, $H \notin \mathcal{F}_\sigma$, so by Theorem 5.6, $H \notin \mathcal{H}(L_0, \mathcal{C}^*)$. On the other hand $f \in L_0$ and $\Delta_h f$ is continuous for any $h \in H$, so f must be essentially continuous. Then for a continuous $\tilde{f}$ we have $f = \tilde{f}$ a.e.

Let $g = \tilde{f} - f$. Then on the one hand $g = 0$ a.e., so also $\Delta_h g = 0$ a.e., on the other hand $\Delta_h g = \Delta_h \tilde{f} - \Delta_h f$ is continuous for any $h \in H$, thus we get that $\Delta_h g = 0$ everywhere for any $h \in H$.

Since f is not continuous, g cannot be equal to zero everywhere, so there exists an x_0 such that $g(x_0) \neq 0$. But $\Delta_h g = 0$ everywhere for any $h \in H$, so for any $y \in H + x_0$ we have $g(y) = g(x_0) \neq 0$. Since $H + x_0$ has positive measure and $g = 0$ a.e. we get a contradiction. ☺

Proposition 6.2. *If $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$, then*

$$\mathcal{H}(\mathcal{C}, \mathcal{G}) \subset \Big\{ \text{subsets of } \mathbf{T} \cap \mathbf{Q} \Big\}.$$

Proof. Assume that there exists a function $f \in \mathcal{C} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}$ for a $h \notin \mathbf{Q}$. If the Fourier series of f is

$$f \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x},$$

then the Fourier series of $\Delta_h f$ is

$$\Delta_h f \sim \sum_{n=-\infty}^{\infty} a_n \left(e^{2\pi inh} - 1\right) e^{2\pi inx}.$$

Therefore, since $h \notin \mathbf{Q}$, we have $Z_f = Z_{\Delta_f}$, where Z_g denotes the set of the indices of the zero terms in the Fourier series of g.

It is well known (see e.g. [E] p. 3) that if $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$, then $f \in \mathcal{G}$ if and only if $Z_f \supset \bigcap\{Z_g : g \in \mathcal{G}\}$. Therefore in our case $\Delta_h f \in G$ and $Z_f = Z_{\Delta_f}$ implies that $f \in \mathcal{G}$, which is a contradiction. ☺

Theorem 6.3. *If $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$ and $\mathcal{G}$ contains the constant functions then $\mathcal{G}$ has the difference property.*

Proof. This follows from Lemma 0.3.1 and Proposition 6.2. ☺

Now we can prove the result we promised earlier in Remark 2.5.

Theorem 6.4. *If $\mathcal{G} \subset \mathcal{F} \subset L_0$ and $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$, then*

$$\mathcal{H}^0(\mathcal{F}^*, \mathcal{G}) = \Big\{\text{ subgroups of } \mathbf{T} \text{ with 0 measure}\Big\},$$
$$\mathcal{H}(\mathcal{F}^*, \mathcal{G}) = \Big\{\text{sets that can be covered by a subgroup of } \mathbf{T} \text{ with 0 measure}\Big\}.$$

Proof. Clearly it is enough to prove the upper equality. We have already proved the inclusion $\supset$ in Proposition 2.4 so now we only need to prove the other inclusion.

By the Monotonicity Lemma, the Triangle-inequality, Theorem 6.1 and Proposition 6.2,

$$\mathcal{H}^0(\mathcal{F}^*, \mathcal{G}) \subset \mathcal{H}^0(L_0, \mathcal{G}) \subset \mathcal{H}(L_0, \mathcal{G}) \subset \mathcal{H}(L_0, \mathcal{C}) \cup \mathcal{H}(\mathcal{C}, \mathcal{G}) =$$
$$= \Big\{\text{sets that can be covered by a subgroup of } \mathbf{T} \text{ with 0 measure}\Big\}. ☺$$

Finally in this section, we prove the result mentioned in Remark 1.8. (We recall that $\mathcal{B}$ is the class of Borel functions on $\mathbf{T}$.)

Theorem 6.5. *We have $H \in \mathcal{H}(\mathcal{B}, \mathcal{C})$ if and only if H can be covered by the set of periods of a proper nonempty Borel subset of $\mathbf{T}$.*

Proof. Suppose that $H \in \mathcal{H}(\mathcal{B}, \mathcal{C})$. Then there exists a function $f \in \mathcal{B} \setminus \mathcal{C}$ such that $\Delta_h f \in \mathcal{C}$ for every $h \in H$. We distinguish between two cases.

Case I. $H \notin \mathcal{F}_\sigma$.
By Theorem 5.6, $\mathcal{H}(L_0, \mathcal{C}^*) \subset \mathcal{F}_\sigma$, so in this case $f \in \mathcal{B} \subset L_0$ and $\Delta_h f \in \mathcal{C} \subset \mathcal{C}^*$ $(h \in H)$ imply that $f \in \mathcal{C}^*$. Then $f = \tilde{f}$ a.e., where $\tilde{f} \in \mathcal{C}$. Let $g = f - \tilde{f}$. Then $g = 0$ a.e. but not everywhere, g is a Borel function and $\Delta_h g = \Delta_h f - \Delta_h \tilde{f}$ is continuous for any $h \in H$.

Thus, since $\Delta_h g = 0$ a.e., necessarily $\Delta_h g = 0$ everywhere. Let $A = \{x : g(x) = 0\}$. Then A is a proper nonempty Borel subset of $\mathbf{T}$ and each $h \in H$ is a period of A.

Case II. $H \in \mathcal{F}_\sigma$.
Let A be a nonempty proper F_σ subgroup of $\mathbf{T}$ that covers A. Then the set of periods of A is $A \supset H$ itself. Therefore the set of the periods of the proper nonempty Borel subset $A \subset \mathbf{T}$ covers H.

Now suppose that H can be covered by the set of periods of a proper nonempty Borel subset of $\mathbf{T}$, say A. Let f be the characteristic function of A. Then, since A is a proper nonempty Borel subset of $\mathbf{T}$, $f \in \mathcal{B} \setminus \mathcal{C}$. On the other hand, for any $h \in A \supset H$ we have $\Delta_h f = 0 \in \mathcal{C}$. Therefore $H \in \mathcal{H}(\mathcal{B}, \mathcal{C})$. ☺

7. Functions with L_∞, and with Lip^1 differences. The construction of Balcerzak, Buczolich and Laczkovich

In this section we prove that $\mathcal{H}(\mathcal{F},\mathcal{G})=\mathcal{F}_\sigma$ if $\mathcal{G}$ is either L_∞, $(\mathrm{Lip}^1)^*$ or Lip^1, and $\mathcal{F}$ is a reasonable class of functions.

We proved the inclusion $\mathcal{H}(\mathcal{F},\mathcal{G})\subset\mathcal{F}_\sigma$ in Section 5 (for these cases), so now we need to prove the other inclusion. That is, for any set $H\in\mathcal{F}_\sigma$ we need to construct a suitable function. We follow the construction of M. Balcerzak, Z. Buczolich and M. Laczkovich ([BBL]).

The following lemma and the following two theorems are essentially proved in [BBL]. The reason, why we repeat their arguments here, is that these results are not stated explicitly in [BBL].

Lemma 7.1. *For any $B\in\mathcal{F}_\sigma$ there exists an infinite nowhere dense closed set A such that kA is also nowhere dense for any $k\in\mathbf{N}$, $A=-A$ and the subgroup of $\mathbf{T}$ generated by A covers B.*

Proof. $B\in\mathcal{F}_\sigma$ means that T has a proper F_σ subgroup C that covers B. Then we can choose nonempty nowhere dense closed sets C_n such that $C=\cup_{n=1}^\infty C_n$. Since $C=-C$, we may assume $C_n=-C_n$. Denoting $D/n=\{x\in\mathbf{T}:|x|<1/n, nx\in D\}$ for every $D\subset\mathbf{T}$, we define $A=\{0\}\cup\bigcup_{n=1}^\infty C_n/n$. Then $A=-A$ and A is an infinite nowhere dense closed set. Thus, for each $k\in\mathbf{N}$, the set kA is a closed subset of $\cup_{n=1}^\infty C/n$, as $kC=C$. Since C is of first category, so is $\cup_{n=1}^\infty C/n$. Therefore kA is a closed set of first category and thus it is nowhere dense. It is clear that the group generated by A contains all C_n's and hence all of $B\subset C$. ☺

Theorem 7.2.

$$\mathcal{H}\left(\bigcap_{0<p<\infty}L_p,\ L_\infty\right)\supset\mathcal{F}_\sigma.$$

Proof. Let $B\in\mathcal{F}_\sigma$. We will construct a function g for B such that

$$g\in\left(\bigcap_{0<p<\infty}L_p\right)\setminus L_\infty\qquad\text{and}\qquad\Delta_h g\in L_\infty\quad\text{for any } h\in B.$$

According to Lemma 7.1, there exists an infinite nowhere dense closed set A such that kA is also nowhere dense for any $k\in\mathbf{N}$, $A=-A$ and the subgroup of $\mathbf{T}$ generated by A covers B.

A lemma of [BBL] (Lemma 1.2) states that if $A \subset \mathbf{T}$ and the closure of kA has measure zero for any $k \in \mathbf{N}$ then there exists a closed set $H \subset \mathbf{T}$ such that H has positive measure and $H + \mathrm{cl}(kA)$ is nowhere dense for any $k \in \mathbf{N}$.

In our case A is closed, so $\mathrm{cl}(kA) = kA$, hence $H + kA$ is nowhere dense for any $k \in \mathbf{N}$. Put $H_{-1} = \emptyset$, $H_0 = H$ and $H_j = H + jA = H_{j-1} + A$ for $j = 1, 2, \ldots$. From $0 \in A$ it follows that $H_{j-1} \subset H_j$. Put $H_\infty = \cup_{j\in\mathbf{N}} H_j$, then $\lim_{j\to\infty} |H_\infty \setminus H_j| = 0$ by $|H_\infty| < \infty$.

Let $j_0 = 1$. If j_{k-1} is defined for a $k \in \mathbf{N}$, choose j_k such that $j_k > j_{k-1}$ and

$$|H_\infty \setminus H_{j_k}| < 1/2^k.$$

For $j_{k-1} < j \le j_k$ we put $c_j = k$. Thus by induction we have defined j_k for all k, and c_j for all j. We put $g(x) = c_j$ if $x \in H_j \setminus H_{j-1}$ ($j \in \mathbf{N}$), and $g(x) = 0$ for $x \in \mathbf{T} \setminus H_\infty$.

First we prove that for any $0 < p < \infty$, $g \in L_p$. Indeed,

$$\int_{\mathbf{T}} |g|^p = \sum_{k=1}^{\infty} \int_{g=k} k^p = \sum_{k=1}^{\infty} |H_{j_k} \setminus H_{j_{k-1}}|\, k^p < \sum_{k=1}^{\infty} k^p/2^{k-1} < \infty.$$

Now we prove that $g \notin L_\infty$. Since A is infinite, $A_\infty = \bigcup_{k\in\mathbf{N}} kA$ is dense in $\mathbf{T}$. Thus for any subinterval J of $\mathbf{T}$, we have $0 < |H_\infty \cap J| = |(H + A_\infty) \cap J|$. Since the H_j's are nowhere dense, there are infinitely many j's for which $|H_j \setminus H_{j-1}| > 0$. Thus, by the definition of g, $g \notin L_\infty$.

Finally we prove that $\Delta_h g \in L_\infty$ for any $h \in B$. Since the subgroup of $\mathbf{T}$ generated by A covers B it is enough to prove this for any $h \in A$.

Let $x \in H_\infty$ and $h \in A$. Then $y = x + h \in H_\infty$, and thus $x \in H_{j_x} \setminus H_{j_x-1}$ and $y \in H_{j_y} \setminus H_{j_y-1}$ with suitable j_x and j_y. If $j_x \le j_y$, then $y = x + h \in H_{j_x} + A = H_{j_x+1}$, and hence $j_y = j_x$ or $j_y = j_x + 1$. Thus, in this case, $|g(y) - g(x)| = 0$ or $|g(y) - g(x)| = |c_{j_x+1} - c_{j_x}| \le 1$. If, on the other hand, $j_x > j_y$ then, using $A = -A$, $x = y - h$, interchanging the roles of x and y, we reach the same conclusion. Therefore, $|g(x + h) - g(x)| \le 1$ holds for any $x \in H_\infty$.

If $x \in \mathbf{T} \setminus H_\infty$ and $h \in A$ then $x + h \in \mathbf{T} \setminus H_\infty$. Indeed, from $x + h \in H_\infty$ it follows that $x + h \in H_j$ for some $j \ge 0$, and then $A = -A$ implies $x = (x + h) - h \in H_j + A \subset H_\infty$, contradicting $x \in \mathbf{T} \setminus H_\infty$. Therefore $|g(x + h) - g(x)| = 0$ holds for any $x \in \mathbf{T} \setminus H_\infty$ and $h \in A$. Thus $|g(x + h) - g(x)| \le 1$ for any $x \in \mathbf{T}$ and $h \in A$. ☺

Theorem 7.3.

$$\mathcal{H}\left(\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha,\ \mathrm{Lip}^1\right) \supset \mathcal{F}_\sigma.$$

Proof. Let $B \in \mathcal{F}_\sigma$. We will construct a function f for B such that $f \in (\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha) \setminus \mathrm{Lip}^1$ and $\Delta_h f \in \mathrm{Lip}^1$ for any $h \in B$.

By Theorem 7.2, there exists a function $g \in (\bigcap_{0<p<\infty} L_p) \setminus L_\infty$ such that $\Delta_h g \in L_\infty$ for any $h \in B$. Let $c = \int_{\mathbf{T}} g$, $g_1(x) = g(x) - c$ and $f(x) = \int_0^x g_1(t)dt$. Then $\int_{\mathbf{T}} g_1 = 0$, so f is a well-defined continuous function on $\mathbf{T}$.

Since $g \in (\bigcap_{0<p<\infty} L_p) \setminus L_\infty$, we have also $g_1 \in (\bigcap_{0<p<\infty} L_p) \setminus L_\infty$, which implies $f \in (\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha) \setminus \mathrm{Lip}^1$. Finally, by the equation

$$|\Delta_h f(x+d) - \Delta_h f(x)| = |\Delta_d f(x+h) - \Delta_d f(x)| = \Big| \int_x^{x+d} \big(g(t+h) - g(t)\big) dt \Big|,$$

$\Delta_h g \in L_\infty$ implies $\Delta_h f \in \mathrm{Lip}^1$ (for any $h \in B$). ☺

Now we can determine the classes of sets of form $\mathcal{H}(\mathcal{F}, L_\infty)$ and $\mathcal{H}(\mathcal{F}, \mathrm{Lip}^1)$ for any reasonable $\mathcal{F}$.

Theorem 7.4. *If* $\bigcap_{0<p<\infty} L_p \subset \mathcal{F} \subset L_0$ *then*

$$\mathcal{H}(\mathcal{F}, L_\infty) = \mathcal{F}_\sigma.$$

In particular for any $0 < p < \infty$

$$\mathcal{H}(L_p, L_\infty) = \mathcal{F}_\sigma.$$

Proof. This is trivial from Theorem 5.5 and Theorem 7.2 by the Monotonicity Lemma. ☺

Theorem 7.5. *If* $\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha \subset \mathcal{F} \subset L_0$ *then*

$$\mathcal{H}\big(\mathcal{F}^*, (\mathrm{Lip}^1)^*\big) = \mathcal{F}_\sigma.$$

If $\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha \subset \mathcal{F} \subset \mathcal{C}$ *then*

$$\mathcal{H}(\mathcal{F}, \mathrm{Lip}^1) = \mathcal{F}_\sigma.$$

In particular for any $0 < \alpha < 1$

$$\mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^1) = \mathcal{F}_\sigma.$$

Proof. The first equality follows from Theorem 5.8 and Theorem 7.3 by the Monotonicity Lemma. Then the second equality follows from the first one by Proposition 2.3. ☺

8. Lip^β differences of Lip^α functions. The class $\mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$

In this section we prove that for any $0 < \alpha < \beta < 1$,

$$p\mathcal{D} \subset \mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) \tag{10}$$

and that all the $\mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ classes are the same.

Lemma 8.1. *If $0 < \gamma < 1$ and (q_n) is an increasing sequence of integers such that $q_{n+1}/q_n > \lambda$ for a suitable $\lambda > 1$, then for any sequence (a_n) of complex numbers,*

$$|a_n| = O(1/q_n^\gamma) \qquad \Longleftrightarrow \qquad g(x) = \sum_{n=1}^{\infty} a_n e^{2\pi i q_n x} \in \mathrm{Lip}^\gamma.$$

Proof.

$\Rightarrow$: Let $k(h) = \max\{k : q_k < 1/h\}$. Then

$$\begin{aligned}
|g(x+h) - g(x)| &= \left| \sum_{n=1}^{\infty} a_n \left(e^{2\pi i q_n h} - 1\right) e^{2\pi i q_n x} \right| = \\
&= \left| \sum_{n=1}^{k(h)} a_n \left(e^{2\pi i q_n h} - 1\right) e^{2\pi i q_n x} + \sum_{n=k(h)+1}^{\infty} a_n \left(e^{2\pi i q_n h} - 1\right) e^{2\pi i q_n x} \right| \le \\
&\le \sum_{n=1}^{k(h)} \frac{C_1}{q_n^\gamma}\, q_n h + \sum_{n=k(h)+1}^{\infty} \frac{C_2}{q_n^\gamma} \le \\
&\le \sum_{n=1}^{k(h)} C_1 \left(\frac{q_{k(h)}}{\lambda^{k(h)-n}}\right)^{1-\gamma} h + \sum_{n=k(h)+1}^{\infty} \frac{C_2}{(q_{k(h)+1}\lambda^{n-k(h)-1})^\gamma} = \\
&= q_{k(h)}^{1-\gamma} h C_1 \sum_{n=1}^{k(h)} \left(\frac{1}{\lambda^{k(h)-n}}\right)^{1-\gamma} + \frac{1}{q_{k(h)+1}^\gamma} \sum_{n=k(h)+1}^{\infty} \frac{C_2}{(\lambda^{n-k(h)-1})^\gamma} \le \\
&\le (1/h)^{1-\gamma} h C_1 \sum_{m=0}^{\infty} \left(\frac{1}{\lambda^m}\right)^{1-\gamma} + \frac{1}{(1/h)^\gamma} \sum_{m=0}^{\infty} \frac{C_2}{(\lambda^m)^\gamma} \le \\
&\le h^\gamma \left(C_1 \sum_{m=0}^{\infty} \frac{1}{(\lambda^{1-\gamma})^m} + \sum_{m=0}^{\infty} C_2 \frac{1}{(\lambda^\gamma)^m} \right) = \\
&= h^\gamma O(1).
\end{aligned}$$

$\Leftarrow$: It is well known that if

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \in \mathrm{Lip}^{\gamma},$$

then $|c_k| = O(|k|^{-\gamma})$. (See e.g. [Z], Vol. I, pp. 46, Th. 4.7.) ☺

We recall that a set $H \subset \mathbf{T}$ is called a *pseudo-Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $n_0(x)$ such that

$$|\sin q_n \pi x| < \varepsilon_n \qquad \text{if } n \geq n_0(x).$$

The family of pseudo-Dirichlet sets are denoted by $p\mathcal{D}$.

Theorem 8.2. *For any $0 < \alpha < \beta < 1$,*

$$p\mathcal{D} \subset \mathcal{H}(\mathrm{Lip}^{\alpha}, \mathrm{Lip}^{\beta}).$$

That is, for any $0 < \alpha < \beta < 1$ and for any pseudo-Dirichlet set H, there exists a Lip^{α} function f such that $\Delta_h f$ is Lip^{β} for any $h \in H$ but f is not Lip^{β}.

Proof. Let H be a pseudo-Dirichlet set. Take a sequence $q_1 < q_2 < \ldots$ and a sequence $\varepsilon_n \to 0$ witnessing the pseudo-Dirichlet property of H. Selecting a suitable subsequence, we may assume that $q_{n+1} > 2q_n$ for every $n \in \mathbf{N}$. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^{\beta} \delta_n} e^{2\pi i q_n x}, \qquad \text{where} \qquad \delta_n = \max(\varepsilon_n, 1/q_n^{\beta-\alpha}).$$

Since $\frac{1}{q_n^{\beta}\delta_n} \leq 1/q_n^{\alpha}$ and $q_{n+1}/q_n > 2$ $(n \in \mathbf{N})$, we can apply Lemma 8.1 to obtain $f \in \mathrm{Lip}^{\alpha}$. On the other hand, since $\delta_n \to 0$, $\frac{1}{q_n^{\beta}\delta_n} \neq O(1/q_n^{\beta})$, so Lemma 8.1 implies that $f \notin \mathrm{Lip}^{\beta}$.

For an $h \in H$,

$$\Delta_h f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^{\beta}\delta_n} \left(e^{2\pi i q_n h} - 1\right) e^{2\pi i q_n x}$$

and

$$\left| \frac{1}{q_n^{\beta}\delta_n} \left(e^{2\pi i q_n h} - 1\right) \right| = \frac{1}{q_n^{\beta}\delta_n} 2|\sin \pi q_n h| \leq \frac{1}{q_n^{\beta}\delta_n} 2\varepsilon_n \leq \frac{2}{q_n^{\beta}},$$

therefore, by Lemma 8.1, $\Delta_h f \in \mathrm{Lip}^{\beta}$. ☺

Corollary 8.3. *For any $0 < \alpha < \beta < 1$,*

$$p\mathcal{D} \subset \mathcal{H}(\mathrm{Lip}^{\alpha}, \mathrm{Lip}^{\beta}) \subset \mathcal{F}_{\sigma}.$$

Proof. This follows from Theorem 5.9 and Theorem 8.2. ☺

Finally we show that it is enough to determine $\mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ for an arbitrary pair of (α, β) with $0 < \alpha < \beta < 1$, namely

$$\mathcal{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathcal{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2})$$

for any $0 < \alpha_1 < \beta_1 < 1$ and $< \alpha_1 < \beta_1 < 1$.

For this we will need the notion of fractional integration. There are several different notions of fractional integrals (see e.g. the monograph [SKM]), here we use the so called Weyl fractional integral which is defined in the following way (see [SKM] p. 263 or [Z] Vol. II p. 133):

Let f be an integrable function on $\mathbf{T}$ and suppose that $\int_{\mathbf{T}} f = 0$. Then, for any $\gamma > 0$ let

$$I_\gamma[f](x) = \int_{\mathbf{T}} f(t) \Psi_\gamma(x - t) dt, \tag{11}$$

where

$$\Psi_\gamma(t) = \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{e^{2\pi i n t}}{(2\pi i n)^\gamma}. \tag{12}$$

It is known (see e.g. [Z]) that the series in (12) converges everywhere on $\mathbf{T} \setminus \{0\}$ and the integral in (11) exists almost everywhere. (If $f \sim \sum c_n e^{2\pi i n x}$ then $I_\gamma[f] \sim \sum c_n \frac{e^{2\pi i n x}}{(2\pi i n)^\gamma}$.)

Since the operator I_γ is defined by a convolution it commutes with the translation operator and it is linear; that is,

$$I_\gamma[f(y + h)](x) = I_\gamma[f(y)](x + h), \tag{13}$$

and

$$I_\gamma[cf + dg] = cI_\gamma[f] + dI_\gamma[g]. \tag{14}$$

Denote by Lip_0^λ the class of the functions of Lip^λ with integral 0 (over $\mathbf{T}$). It is also well known (see e.g. [SKM] p. 275) that if $\gamma, \lambda > 0$ and $\gamma + \lambda < 1$, then I_γ is a bijection (actually it is an isomorphism) between the classes Lip_0^λ and $\mathrm{Lip}_0^{\lambda+\gamma}$; that is,

$$I_\gamma : \ \mathrm{Lip}_0^\lambda \leftrightarrow \mathrm{Lip}_0^{\lambda+\gamma} \qquad (\lambda + \gamma < 1). \tag{15}$$

Theorem 8.4. *For any* $0 < \alpha_1 < \beta_1 < 1$ and $0 < \alpha_2 < \beta_2 < 1$,

$$\mathcal{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathcal{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2}).$$

Proof. First we prove that if $0 < \alpha < \beta$ and $\beta + \gamma < 1$ then

$$\mathcal{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma}) = \mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta). \tag{16}$$

Indeed, (13) and (14) implies that the operator I_γ also commutes with the difference operator Δ_h; that is

$$\Delta_h I_\gamma[f] = I_\gamma[\Delta_h f]. \tag{17}$$

It follows from (15) and (17) that if $f_0 \in \mathrm{Lip}_0^\alpha \setminus \mathrm{Lip}_0^\beta$ and $\Delta_h f_0 \in \mathrm{Lip}_0^\beta$ for every $h \in H$ then $I_\gamma[f_0] \in \mathrm{Lip}_0^{\alpha+\gamma} \setminus \mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h I_\gamma[f_0] \in \mathrm{Lip}_0^{\beta+\gamma}$ for every $h \in H$. Furthermore, if $g_0 \in \mathrm{Lip}_0^{\alpha+\gamma} \setminus \mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h g_0 \in \mathrm{Lip}_0^{\beta+\gamma}$ for every $h \in H$ then $I_\gamma^{-1}[g_0] \in \mathrm{Lip}_0^\alpha \setminus \mathrm{Lip}_0^\beta$ and $\Delta_h I_\gamma^{-1}[g_0] \in \mathrm{Lip}_0^\beta$ for every $h \in H$. Therefore if the function $f : \mathbf{T} \to \mathbf{R}$ witnesses that $H \in \mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ then $I_\gamma[f_0]$ - where $f_0 = f - \int_{\mathbf{T}} f$ - witnesses that $H \in \mathcal{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma})$; furthermore if the function $g : \mathbf{T} \to \mathbf{R}$ witnesses that $H \in \mathcal{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma})$ then $I_\gamma^{-1}[g_0]$ - where $g_0(x) = g - \int_{\mathbf{T}} g$ - witnesses that $H \in \mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$.

Now we prove that for any $0 < \eta < \delta < \beta < 1$,

$$\mathcal{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^\beta) = \mathcal{H}(\mathrm{Lip}^{\beta-\eta}, \mathrm{Lip}^\beta). \tag{18}$$

Indeed, by (17), $\mathcal{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta-\delta/2}) = \mathcal{H}(\mathrm{Lip}^{\beta-\delta/2}, \mathrm{Lip}^\beta)$, which implies, by Lemma 0.3.5, that $\mathcal{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^\beta) = \mathcal{H}(\mathrm{Lip}^{\beta-\delta/2}, \mathrm{Lip}^\beta)$. Thus we have also $\mathcal{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^\beta) = \mathcal{H}(\mathrm{Lip}^{\beta-\delta/2^k}, \mathrm{Lip}^\beta)$ for any $k \in \mathbf{N}$. Then, by the Monotonicity Lemma, $\mathcal{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^\beta) = \mathcal{H}(\mathrm{Lip}^{\beta-\eta}, \mathrm{Lip}^\beta)$.

Finally, supposing that $\beta_1 \le \beta_2$ and applying (17) and (18), we get

$$\mathcal{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathcal{H}(\mathrm{Lip}^{\alpha_1+\beta_2-\beta_1}, \mathrm{Lip}^{\beta_2}) = \mathcal{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2}). \quad ☺$$

Remark 8.5. Unfortunately this proof does not work if β_1 or β_2 equals 1. Namely, (15) is not true for $\lambda + \gamma = 1$. In this case I_γ is a bijection between $\mathrm{Lip}_0^{1-\gamma}$ and Λ_{*0}, the class of Zygmund functions on $\mathbf{T}$ with 0 integral.

(A function f is Zygmund, if for any x and h, $|f(x+h) - 2f(x) + f(x-h)| \le Ch$. The class of Zygmund functions is denoted by Λ_*. It is known (see e.g. [Z] Vol. I p. 43-44 and Vol. II p. 138) that

$$\mathrm{Lip}^1(\mathbf{R}) \subset \Lambda_* \subset \mathrm{Lip}^\alpha(\mathbf{R}) \quad \forall 0 < \alpha < 1,$$

and $\Lambda_* \ne \mathrm{Lip}^1(\mathbf{R})$.)

Therefore with this method we can only prove that

$$\mathcal{H}(\mathrm{Lip}^\alpha, \Lambda_*(\mathbf{T})) = \mathcal{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1})$$

for any $0 < \alpha < 1$ and $0 < \alpha_1 < \beta_1 < 1$.

However, if one can find a linear operator I that commutes with the translation operator and I is a bijection between Lip^1 and Lip^α (for a fixed $0 < \alpha < 1$) or Λ_* then Theorem 8.4. would remain true for $\beta_i = 1$. Then, according to Theorem 7.5, it would mean that $\mathcal{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) = \mathcal{F}_\sigma$ for any $0 < \alpha < \beta \le 1$.

9. L_q differences of L_p functions. The class $\mathcal{H}(L_p, L_q)$

In this section we prove that $\mathcal{H}(L_p, L_q) \supset p\mathcal{D}$. We also investigate the possible improvement of our earlier result, $\mathcal{H}(L_p, L_q) \subset \mathcal{F}_\sigma$.

Lemma 9.1. *Suppose that $0 < p < q < \infty$ and (a_n) is a sequence of positive reals such that*

$$\sum_{j=1}^{\infty} a_j < \infty \qquad \text{and} \qquad \sum_{j=k}^{\infty} a_j \geq C/k^N \quad \text{for fixed } C > 0 \text{ and } N \geq 2.$$

Then there exists a sequence of positive reals (c_j) such that

$$\text{(A)} \qquad \sum_{j=1}^{\infty} a_j c_j^p < \infty,$$

$$\text{(B)} \qquad \sum_{j=1}^{\infty} a_j c_j^q = \infty,$$

and

$$\text{(C)} \qquad \sum_{j=1}^{\infty} a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|) \big)^q < \infty.$$

Proof. First we define a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ such that

$$\text{(i)} \qquad \sum_{j=n_k}^{\infty} a_j \geq C'/k^N \qquad \text{for infinitely many } k \text{ for a fixed } C' > 0 \quad \text{and}$$

$$\text{(ii)} \qquad \sum_{j=n_k+1}^{\infty} a_j < C/k^N \qquad \text{for every } k \in \mathbf{N}.$$

Let $k \in \mathbf{N}$ and suppose that n_{k-1} is already defined. If $\sum_{j=n_{k-1}+1}^{\infty} a_j \geq C/k^N$ then let

$$n_k = \max\Big\{ \, i \; : \; \sum_{j=i}^{\infty} a_j \geq C/k^N \, \Big\}.$$

Otherwise let $n_k = n_{k-1}+1$. Then clearly $n_k > n_{k-1}$ and (ii) holds. If $\sum_{j=n_{k-1}+1}^{\infty} a_j \geq C/k^N$ for infinitely many k then (i) clearly holds with $C' = C$. Otherwise, there exists an $m \in \mathbf{N}$ such that $n_k = n_m + (k-m)$ for every $k \geq m$. Then for every $k \geq m$,

$$\sum_{j=n_k}^{\infty} a_j = \sum_{j=n_m+k-m}^{\infty} a_j \geq \frac{C}{(n_m+k-m)^N} \geq \frac{C}{((n_m+1-m)k)^N},$$

thus (i) holds for $C' = \frac{C}{(n_m+1-m)^N}$.

Let

$$b_k = a_{n_k+1} + a_{n_k+2} + \ldots + a_{n_{k+1}} \qquad (k = 0, 1, 2, \ldots),$$

and

$$\alpha = \sup\Big\{\ \beta\ : \sum_{k=1}^{\infty} b_k k^{\beta} < \infty\ \Big\}.$$

We claim that $1 \leq \alpha \leq N$.

$1 \leq \alpha$: By (ii),

$$\sum_{k=1}^{\infty} b_k k = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} b_n = \sum_{k=1}^{\infty}\sum_{j=n_k+1}^{\infty} a_j < \sum_{k=1}^{\infty} C/k^N < \infty.$$

$\alpha \leq N$: Let $\varepsilon > 0$ arbitrary. If (i) holds for $k \geq 2$ then

$$\sum_{n=1}^{\infty} b_n n^{N+\varepsilon} \geq \sum_{n=k-1}^{\infty} b_n n^{N+\varepsilon} \geq \sum_{n=k-1}^{\infty} b_n (k-1)^{N+\varepsilon} =$$

$$= (k-1)^{N+\varepsilon} \sum_{j=n_{k-1}+1}^{\infty} a_j \geq (k-1)^{N+\varepsilon} \sum_{j=n_k}^{\infty} a_j \geq (k-1)^{N+\varepsilon} \frac{C'}{k^N}.$$

Since k can be arbitrarily big this implies that $\sum_{n=1}^{\infty} b_n n^{N+\varepsilon} = \infty$, thus $\alpha \leq N$.

Choose γ such that $\alpha/q < \gamma < \min(\alpha/p, (\alpha/q)+1)$, and let $c_n = k^{\gamma}$ for any $n_k < n \leq n_{k+1}$ $(k = 0, 1, \ldots)$. Then

$$\sum_{j=2}^{\infty} a_j c_j^p = \sum_{k=0}^{\infty} b_k (k^{\gamma})^p < \infty,$$

since $\gamma p < \alpha$.

$$\sum_{j=2}^{\infty} a_j c_j^q = \sum_{k=0}^{\infty} b_k (k^{\gamma})^q = \infty,$$

since $\gamma q > \alpha$.

Finally, applying the mean value theorem, we have

$$\max(|k^{\gamma} - (k-1)^{\gamma}|, |(k+1)^{\gamma} - k^{\gamma}|) \leq \gamma(k+1)^{\gamma-1} \leq \gamma(2k)^{\gamma-1},$$

thus

$$\sum_{j=1}^{\infty} a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|) \big)^q \le \sum_{k=0}^{\infty} b_k \big(\max(|k^\gamma - (k-1)^\gamma|, |(k+1)^\gamma - k^\gamma|) \big)^q \le$$

$$\le \sum_{k=0}^{\infty} b_k \left(\gamma (2k)^{\gamma-1} \right)^q = (2^{\gamma-1}\gamma)^q \sum_{k=0}^{\infty} b_k k^{q(\gamma-1)} < \infty,$$

since $q(\gamma - 1) < \alpha$. ☺

Lemma 9.2. *Suppose that* A and H *are closed subsets of* $\mathbf{T}$, $A = -A$, $0 \in A$, H *has positive measure and there exist constants* $C > 0$ and $N \ge 2$ *such that*

$$|H + kA| \le 1 - C/k^N \qquad (\forall k \in \mathbf{N}). \tag{19}$$

Then $A \in \mathcal{H}(L_p, L_q)$ *for any* $0 < p < q < \infty$.

Proof. For given $0 < p < q < \infty$ we construct a function $g \in L_p \setminus L_q$ such that $\Delta_h g \in L_q$ for any $h \in A$. The construction is a modification of the construction of Theorem 7.2; that is, we use the method in [BBL] again. Using the same notation, let $H_j = H + jA$, $H_\infty = \cup_{j \in \mathbf{N}} H_j$. If A is a finite subset of $\mathbf{Q} \cap \mathbf{T}$ then an arbitrary periodic function $g \in L_p \setminus L_q$ with period $1/m$ (where m is a common denominator of the elements of A) satisfy the conditions. Otherwise, since $A = -A$, H_∞ has infinitely many periods; thus $|H| > 0$ implies $|H_\infty| = 1$.

Let $a_j = |H_j \setminus H_{j-1}|$ $(j \in \mathbf{N})$. Then, using $|H_\infty| = 1$ and (19),

$$\sum_{n=k}^{\infty} a_n = |H_\infty \setminus H_{k-1}| = |\mathbf{T} \setminus H_{k-1}| \ge |\mathbf{T} \setminus H_k| \ge C/k^N.$$

Then, applying Lemma 9.1, there exists a sequence of positive reals (c_j) such that (A), (B) and (C) holds.

Like in the proof of Theorem 7.2, let $g(x) = c_j$ if $x \in H_j \setminus H_{j-1}$ $(j \in \mathbf{N})$, and $g(x) = 0$ for $x \in \mathbf{T} \setminus H_\infty$. Then, using (A) and (B),

$$\int_{\mathbf{T}} |g|^p = \sum_{j=1}^{\infty} a_j c_j^p < \infty \quad \text{and} \quad \int_{\mathbf{T}} |g|^q = \sum_{j=1}^{\infty} a_j c_j^q = \infty,$$

therefore $g \in L_p \setminus L_q$.

Similarly like in the proof of Theorem 7.2, if $h \in A$, $x \in H_{j_x} \setminus H_{j_x - 1}$ and $y = x + h \in H_{j_y} \setminus H_{j_y - 1}$ then $|j_y - j_x| \le 1$. Thus $|f(x+h) - f(x)| \le \max(|c_{j_x} - c_{j_x - 1}|, |c_{j_x + 1} - c_{j_x}|)$. Hence, using (C), for any $h \in A$,

$$\int_{\mathbf{T}} |\Delta_h g|^q \le \sum_{j=1}^{\infty} a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|)^q \big) < \infty.$$

Therefore $\Delta_h g \in L_q$. ☺

We recall that a set $H \subset \mathbf{T}$ is a *Dirichlet set* if there exists an increasing sequence of integers (q_n) such that $|\sin q_n \pi x| \to 0$ uniformly on H; that is, there exists a sequence (ε_n) converging to zero such that for any $x \in H$

$$|\sin q_n \pi x| < \varepsilon_n \qquad (\ \forall\ n \in \mathbf{N}\ ).$$

A set $H \subset \mathbf{T}$ is a *pseudo-Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $k(x)$ such that

$$|\sin q_n \pi x| < \varepsilon_n \qquad \text{if} \quad n > k(x).$$

The family of Dirichlet sets and pseudo-Dirichlet sets are denoted by $\mathcal{D}$ and $p\mathcal{D}$, resp.

The following lemma was proved by Géza Kós ([Ko]).

Lemma 9.3. *For any pseudo-Dirichlet set $H \subset \mathbf{T}$ there exists a Dirichlet set $\Lambda \subset \mathbf{T}$ such that the group generated by Λ contains H.*

Proof. Take sequences $q_1 < q_2 < \dots$ and $\varepsilon_n \to 0$ and a function $k : H \to \mathbf{N}$ witnessing the pseudo-Dirichlet property of H. Taking a suitable subsequence we can assume that $\varepsilon_n = \frac{1}{n}$. Then, denoting $|\sin \pi x|$ by $\|x\|$, we have $\|q_n x\| < \frac{1}{n}$ for any $x \in H$ and $n > k(x)$.

First we show that the sequence $q_1, q_2, \ldots$ can be replaced by a sequence $r_1, r_2, \ldots$ such that (i) for $n > k(x)$ we still have $\|r_n x\| < \frac{1}{n}$, (ii) each r_n divides r_{n+1}, and (iii) $r_{n+1} \geq 2(n+1) r_n$ $(n \in \mathbf{N})$. We define the sequence (r_n) by induction. Let

$$r_1 = q_1 \quad \text{and} \quad r_{n+1} = 2(n+1) r_n q_{2(n+1)^2 r_n}.$$

Then clearly r_n divides r_{n+1} and $r_{n+1} \geq 2(n+1) r_n$ $(n \in \mathbf{N})$. For $n+1 > k(x)$ we have

$$\|r_{n+1} x\| = \|2(n+1) r_n q_{2(n+1)^2 r_n} x\| \leq$$

$$\leq 2(n+1) r_n \|q_{2(n+1)^2 r_n} x\| < 2(n+1) r_n \frac{1}{2(n+1)^2 r_n} = \frac{1}{n+1}.$$

Now we can define Λ-t. Let

$$\Lambda = \left\{ x \in \mathbf{T} :\ \forall n\ \|r_n x\| < \frac{\pi}{n} \right\}.$$

It is clear that Λ is a Dirichlet set. We need to show that any element of H can be written as a finite sum of elements in Λ.

Let $x \in H$ and $m > k(x)$. Clearly x can be written in the form of $x = \frac{a}{r_m} + y$, where $a \in \mathbf{Z}$ and $\|y\| \leq \frac{\pi}{2r_m}$. We have $y \in \Lambda$, since if $n \geq m$ then

$$\|r_n y\| = \left\| r_n \left(x - \frac{a}{r_m} \right) \right\| \leq \|r_n x\| + \left\| a \frac{r_n}{r_m} \right\| = \|r_n x\| < \frac{1}{n},$$

and if $n < m$ then

$$\|r_n y\| \leq r_n \|y\| \leq r_{m-1} \frac{\pi}{2r_m} \leq r_{m-1} \frac{\pi}{4mr_{m-1}} = \frac{\pi}{4m} < \frac{1}{n}.$$

On the other hand $\frac{1}{r_m} \in \Lambda$, too, since if $n \geq m$ then $\left\| r_n \frac{1}{r_m} \right\| = 0$, and if $n < m$ then

$$0 < r_n \frac{1}{r_m} \leq \frac{r_{m-1}}{r_m} \leq \frac{1}{2m} < \frac{1}{n}.$$

Thus x is indeed in the subgroup generated by Λ. ☺

Theorem 9.4. *For any $0 < p < q < \infty$,*

$$\mathcal{H}(L_p, L_q) \supset p\mathcal{D}.$$

That is, for any pseudo-Dirichlet set H there exists a function $g \in L_p \setminus L_q$ such that $\Delta_h g \in L_q$ for any $h \in H$.

Proof. By Lemma 9.3, there exists a Dirichlet set $\Lambda \subset \mathbf{T}$ such that the group generated by Λ contains H. Then clearly it is enough to prove $\Delta_h g \in L_q$ for any $h \in \Lambda$.

Take a sequence $q_1 < q_2 < \ldots$ and a sequence $\varepsilon_n \to 0$ witnessing the Dirichlet property of Λ. Then any subsequence of (q_n, ε_n) also witnesses the Dirichlet property of Λ, so we can assume that $\varepsilon_n < C/2n^3$, where $C < 1/(\sum_{n=1}^{\infty} 2/n^2)$ $(= 3/\pi^2)$ is fixed.

Let

$$A = \left\{ \alpha \in \mathbf{T} : |\sin q_n \pi \alpha| \leq \varepsilon_n \ \forall n \in \mathbf{N} \right\}.$$

Then clearly $0 \in A$, $A = -A$, A is closed and $\Lambda \subset A$. Thus, according to Lemma 9.2, it is enough to find a closed set $H \subset \mathbf{T}$ with positive measure with the property (19).

Denoting $\mathbf{Z}/m\mathbf{Z}$ by $\mathbf{Z}_m$, let

$$B_n = \bigcup_{j \in \mathbf{Z}_{q_n}} S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} \right) \qquad (n \in \mathbf{N}),$$

where $S(x, r)$ denotes the neighborhood of x with radius r. Let

$$B = \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad H = \mathbf{T} \setminus B.$$

Then H is clearly closed. In addition,

$$|B| \leq \sum_{n=1}^{\infty} |B_n| \leq \sum_{n=1}^{\infty} q_n 2 \frac{C}{n^2 q_n} = C \sum_{n=1}^{\infty} 2/n^2 < 1,$$

thus $|H| > 0$. Therefore we only need to prove (19).

We claim that if $|q_n\beta| < \varepsilon$ then

$$B_n + \beta \supset \bigcup_{j\in \mathbf{Z}_{q_n}} S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} - \frac{\varepsilon}{q_n}\right). \tag{20}$$

Indeed, since (on $\mathbf{T}$) $\beta = p_{n,\beta}/q_n + (q_n\beta)/q_n$ (for a proper $p_{n,\beta} \in \mathbf{Z}_{q_n}$),

$$S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} - \frac{\varepsilon}{q_n}\right) \subset S\left(\frac{j - p_{n,\beta}}{q_n}, \frac{C}{n^2 q_n}\right) + \beta.$$

For any $\alpha \in A$ we have $|q_k\alpha| \le |\sin(\pi q_k \alpha)| \le \varepsilon_k < C/2k^3$. Hence, if $\alpha_1, \dots, \alpha_k \in A$, then $|q_k(\alpha_1 + \ldots + \alpha_k)| \le kC/2k^3 = C/2k^2$. Therefore, by (20), for any $\beta \in kA$,

$$B + \beta \supset B_k + \beta \supset \bigcup_{j\in \mathbf{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{k^2 q_k} - \frac{(C/2k^2)}{q_k}\right) = \bigcup_{j\in \mathbf{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right).$$

Thus, for any $\beta \in kA$,

$$H + \beta \subset \mathbf{T} \setminus \bigcup_{j\in \mathbf{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right).$$

Therefore

$$H + kA \subset \mathbf{T} \setminus \bigcup_{j\in \mathbf{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right),$$

so

$$|H + kA| \le 1 - q_k 2\frac{C}{2k^2 q_k} = 1 - C/k^2. \quad ☺$$

Corollary 9.5. *For any $0 < p < q < \infty$,*

$$p\mathcal{D} \subset \mathcal{H}(L_p, L_q) \subset \mathcal{F}_\sigma.$$

Proof. This follows from Theorem 5.9 and Theorem 9.4. ☺

Unfortunately we cannot prove that $\mathcal{H}(L_p, L_q)$ is contained in a strictly smaller class than $\mathcal{F}_\sigma$. However, the following theorem suggests the conjecture that $\mathcal{H}(L_p, L_q) \subset \mathcal{N}$. The proof is based on the "ejectivity" property of the compact non-N-sets (see [LRu]). Maybe in a similar way one can prove that $\mathcal{H}(L_p, L_q) \subset \mathcal{N}$.

For the proof we need a further class of thin sets in harmonic analysis.

Notation 9.6. A Borel set $F \subset \mathbf{T}$ is called a *weak Dirichlet set* (see e.g. in [HMP] p. 48), if for every probability measure μ supported by F,

$$\limsup_{|n|\to\infty} |\hat{\mu}(n)| = 1, \qquad \text{where} \quad \hat{\mu}(n) = \int_{\mathbf{T}} e^{2\pi i n t} d\mu(t).$$

Theorem 9.7. *If $H \subset \mathbf{T}$ is not an N-set, $f : \mathbf{T} \to \mathbf{R}$ is a measurable function, and $\Delta_h f \in L_\infty$ for any $h \in H$ then $f \in L_p$ for any $p > 0$.*

Proof. Let

$$K_m = \{h : |\Delta_h f| \le m \text{ a. e.}\} \qquad (m \in \mathbf{N}).$$

Then clearly $H \subset \cup_{m\in\mathbf{N}} K_m$, so $H \notin \mathcal{N}$ implies $\cup_{m\in\mathbf{N}} K_m \notin \mathcal{N}$. As we saw in the proof of Proposition 5.1, K_m is closed, thus (K_m) is an increasing sequence of compact sets.

It is known (see e.g. in [Ka] p. 190) that for each increasing sequence (K_m) of compact weak Dirichlet sets, $\cup_{m\in\mathbf{N}} K_m \in \mathcal{N}$. Therefore, in our case, there exists an $m \in \mathbf{N}$ such that K_m is not a weak Dirichlet set. Dividing f by m, we can assume that $m = 1$. Therefore, denoting K_1 by K,

$$K = \{h : |\Delta_h f| \le 1 \text{ a. e.}\}$$

and there exists a probability measure μ supported by K such that

$$\limsup_{|n|\to\infty} |\hat{\mu}(n)| < 1.$$

Thus there exists an $\eta > 0$ such that $\mathrm{Re}\ \hat{\mu}(n) \le 1 - \eta$ for every $n \in \mathbf{Z}$ with at most finitely many exceptions. If, for any $n \ne 0$, $\mathrm{Re}\ \hat{\mu}(n) = 1$ then $e^{2\pi int} = 1$ μ-a.e., so for any $k \in \mathbf{Z}$ also $e^{2\pi inkt} = 1$ μ-a.e., which is impossible, since $\limsup_{|n|\to\infty} |\hat{\mu}(n)| < 1$. Therefore we can assume, with a suitable $\eta > 0$, that

$$\mathrm{Re}\ \hat{\mu}(n) \le 1 - \eta \qquad (\forall n \in \mathbf{Z} \setminus \{0\}).$$

It is proved in [LRu] that if there exists a probability measure μ supported on K such that $\mathrm{Re}\ \hat{\mu}(n) \le 1 - \eta$ for every $n \in \mathbf{Z} \setminus \{0\}$ then K is "essentially ejective", which means that for every $x \in (0, 1]$,

$$\zeta_K(x) \ge \eta x(1-x), \qquad \text{where} \quad \zeta_K(x) = \inf_{|A|=x} \sup_{h\in K} |(A+h) \setminus A|.$$

Therefore in our case $\zeta_K(x) \ge \eta x(1-x)$, thus

$$\sup_{h\in K} |(A+h) \setminus A| \ge \eta\ |A|\ (1 - |A|) \tag{21}$$

for any $A \subset \mathbf{T}$ with $|A| > 0$.

Now we define a sequence $A_m, A_{m+1}, \dots$ of subsets of $\mathbf{T}$ by induction. Since f is measurable there exists an $n_0 \in \mathbf{N}$ such that

$$A_{n_0} = \{x \in \mathbf{T} : |f(x)| < n_0\}$$

has positive measure. Assume that A_n is already defined ($n \ge n_0$). By (21), there exists a $h_n \in K$ such that

$$|(A_n + h_n) \setminus A_n| \ge \frac{\eta}{2}\ |A_n|\ (1 - |A_n|). \tag{22}$$

Then let $A_{n+1} = A_n \cup (A_n + h_n)$.

Let

$$C_n = \{x \in \mathbf{T} : |f(x)| \geq n\} \quad \text{and} \quad c_n = |C_n| \quad (n = 0, 1, \ldots).$$

By the definition of K, it is easy to see by induction that $|f(x)| < n$ for a. e. $x \in A_n$, which means that $c_n \leq |\mathbf{T} \setminus A_n|$ $(n \geq n_0)$. Using the notation $b_n = |\mathbf{T} \setminus A_n|$, from (22) we get

$$b_n - b_{n+1} \geq \frac{\eta}{2}(1 - b_n)b_n \geq \frac{\eta}{2}(1 - b_{n_0})b_n \qquad (n \geq n_0),$$

thus

$$b_{n+1} \leq b_n \left(1 - \frac{\eta}{2}(1 - b_{n_0})\right) \qquad (n \geq n_0).$$

Therefore, denoting $1 - \frac{\eta}{2}(1 - b_{n_0})$ by λ,

$$b_n \leq b_{n_0} \lambda^{n - n_0} \qquad (n \geq n_0).$$

Since $\eta > 0$ and $1 - b_{n_0} = |A_{n_0}| > 0$ we have $\lambda < 1$.

Let $p > 0$. Then

$$\begin{aligned}
\int_{\mathbf{T}} |f|^p &= \sum_{n=1}^{\infty} \int_{C_{n-1} \setminus C_n} |f|^p \leq \\
&\leq \sum_{n=1}^{\infty} (c_{n-1} - c_n) n^p = \\
&= \sum_{m=0}^{\infty} c_m \left((m+1)^p - m^p\right) \leq \\
&\leq O(1) + \sum_{m=n_0}^{\infty} b_m \left((m+1)^p - m^p\right) \leq \\
&\leq O(1) + \sum_{m=n_0}^{\infty} b_{n_0} \lambda^{m - n_0} (m+1)^p < \\
&< \infty. \quad ☺
\end{aligned}$$

Remark 9.8. As we remarked earlier this proof is based on the "ejectivity" property of a compact not N-set. On the other hand the proof of Theorem 9.4 uses the "non-ejectivity" of a pseudo-Dirichlet set: the argument of the proof of Theorem 9.7 shows that the condition (19) of Lemma 9.2 cannot be satisfied if H is essentially ejective. Since in the class of F_σ sets the non-essentially ejective sets are the N-sets (see [LRu]), this motivates the following conjecture:

Conjecture. For any $0 < p < q < \infty$,

$$\mathcal{H}(L_p, L_q) = \mathcal{N}.$$

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