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A 1-dimensional subset of the reals that intersects each of its translates in at most a single point

Abstract

We construct a compact subset of $\mathbf{R}$ with Hausdorff dimension 1 that intersects each of its non-identical translates in at most one point. Moreover, one can make the set to be linearly independent over the rationals.

In 1984 P. Mattila [2] constructed compact subsets A and B of $\mathbf{R}$ with Hausdorff dimension 1 such that the intersection of A and any translate of B contains at most one point. In this note we show that - if we allow only non-identical translations - one can also have $A = B$.

We call a set of 3 or 4 real numbers $x_1 < x_2 \leq x_3 < x_4$ a *rectangle* if $x_2 - x_1 = x_4 - x_3$.

Note that a set intersects each of its translates in at most one point if and only if the set does not contain a rectangle. (Here and in the sequel by set we will always mean a subset of $\mathbf{R}$ and by translate a non-identical translate.)

Theorem 1 *There exists a compact set with Hausdorff dimension 1 that intersects each of its translates in at most one point.*

Proof. Let $\delta_m = 1/(6^{m-1}m!)$. We define inductively compact sets A_m as disjoint unions of the closed intervals $[n_{i_1\ldots i_m}\delta_m, (n_{i_1\ldots i_m}+1)\delta_m]$ for $1 \leq i_k \leq k$, $1 \leq k \leq m$. We will denote by $I_1^m, I_2^m, \ldots, I_{m!}^m$ the intervals of A_m and by $(J_1, J_2, \ldots)$ the sequence $(I_1^1, I_1^2, I_2^2, I_1^3, \ldots, I_{3!}^3, \ldots)$.

Let $n_1 = 0$. (Then $A_1 = I_1^1 = J_1 = [0,1]$.) Assume that $A_1, \ldots, A_m$ have already been defined. If $n_{i_1\ldots i_m}\delta_m \notin J_m$ then let

$$n_{i_1\ldots i_m i} = 6(m+1)n_{i_1\ldots i_m} + 6i - 6 \qquad (i = 1, \ldots, m+1), \tag{1}$$

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and if $n_{i_1\dots i_m}\delta_m \in J_m$ then let

$$n_{i_1\dots i_m i} = 6(m+1)n_{i_1\dots i_m} + 6i - 3 \qquad (i = 1,\dots,m+1). \tag{2}$$

Thus

$$[n_{i_1\dots i_m i}\delta_{m+1}, (n_{i_1\dots i_m i}+1)\delta_{m+1}] \subset [n_{i_1\dots i_m}\delta_m, (n_{i_1\dots i_m}+1)\delta_m]$$

for $i = 1,\dots,m+1$, which means that the intervals of A_{m+1} are contained in the intervals of A_m.

Let $A = \cap_{l=1}^{\infty} A_m$. Then A has Hausdorff dimension 1, cf. [1] Example 4.6. Hence, by our previous remark, it is enough to show that A does not contain a rectangle.

Let $x_1 < x_2 \le x_3 < x_4$ be points of A. Take an m such that $\delta_m < x_2 - x_1$. Then if $x_1 \in I_j^m = J_M$ then none of x_2, x_3 and x_4 is in I_j^m. Thus, when we defined A_{M+1}, we used (2) for defining the interval that contains x_1 and (1) for defining the intervals that contain x_2, x_3 and x_4. This implies that x_1 is of the form $(6N_1+3)\delta_M + \varepsilon_1$ but x_2, x_3 and x_4 are of the form $6N_j\delta_M + \varepsilon_j$, where $N_1,\dots,N_4$ are integers and $0 \le \varepsilon_i \le \delta_M$ for $i = 1,\dots,4$. Thus $x_2 - x_1 \ne x_4 - x_3$, which means that (x_1, x_2, x_3, x_4) is not a rectangle. □

Remark 2 Slightly modifying the above construction (by replacing 6 with a slowly increasing sequence of even numbers) one can also get a compact set with Hausdorff dimension 1 which is linearly independent over the rationals. (The existence of a linearly independent perfect set is well known, even in any non-discrete locally compact abelian group, see e. g. [3].)

References

[1] K. Falconer, *Fractal Geometry*, John Wiley & Sons, 1990.

[2] P. Mattila, Hausdorff dimension and capacities of intersections of sets in n-space, *Acta Math.*, **152** (1984), 77-105.

[3] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York - London, 1962.