THE MOUNTAIN CLIMBERS' PROBLEM

BY

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ABSTRACT. We show that two climbers can climb a mountain in such a way that at each moment they are at the same height above the sea level, supposing that the mountain has no plateau. That is, if f and g are continuous functions mapping $[0,1]$ to $[0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, and neither f nor g has an interval of constancy then there exist continuous functions k and h: $[0,1] \to [0,1]$ satisfying $k(0) = h(0) = 0$, $k(1) = h(1) = 1$ and $f \circ k = g \circ h$.

Introduction. In this paper we shall examine the following problem:
Two mountain climbers begin at sea level, at opposite ends of a (two-dimensional) chain of mountains. Can they find routes along which to travel, always maintaining equal altitudes, until they eventually meet?

If we now select a point of maximum altitude and reparametrize, we can formulate it as follows:

$(*)$ Let f and g be continuous functions mapping $[0,1]$ to $[0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Are there continuous functions k and h: $[0,1] \to [0,1]$ satisfying $k(0) = h(0) = 0$, $k(1) = h(1) = 1$ and $f \circ k = g \circ h$?

This problem, in a slightly different form, was posed by J. V. Whittaker in [2]. Whittaker proves that the answer is "*yes*" if f and g are piecewise monotone (see also [1]). He also shows that $(*)$ is not true in general. Namely, it is easy to verify that for the following two functions there are no corresponding k and h: let f be a monotone function which is constant in an interval, let g be a function which oscillates around this value. (See fig.1.)

It does not follow from this counter-example that a "typical continuous" function is not climbable: f is a very special continuous function (it has an interval of constancy), and

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this is the reason that the pair of functions (f, g) is a counter- example. We can hope that all the counter-examples are similary restricted, namely that the answer for $(*)$ is "*yes*" if we suppose that neither f nor g have an interval of constancy.

This is the main result of this paper.

Notation 1.

$\mathcal{F} = \{f | f : [0,1] \to [0,1] \text{ continuous}, \ f(0) = 0, \ f(1) = 1\}$.

Let $\mathbf{C}[a,b]$ denote the set of continuous functions defined in the interval $[a,b]$ where $a < b$.

$[a,b]$ will denote the closed interval $[a,b]$ if $a < b$; $[b,a]$ if $b < a$ and it will denote the point a if $a = b$.

$\mathcal{G} = \{f \in \mathbf{C}[a,b] \mid a, b \in \mathbf{R}, \ a < b, \ f([a,b]) = [f(a), f(b)]\}$.

$f \bowtie g$ (f and g *match each other*), if $f, g \in \mathcal{G}$ and their range is the same.

$f \sim g$ (f and g are *climbable*) if $f \bowtie g$ and $\exists k, h \in \mathcal{G} : f \circ k = g \circ h$, and the ranges of k and h are equal to the domains of definition of f and g, respectively.

By "*monotone*" we shall mean non-decreasing or non- increasing.

Remarks

- $f, g \in \mathcal{F} \Rightarrow f \bowtie g$
- If $f \bowtie g$, then by linear change of parameter we can get two functions f_1, g_1 from $\mathcal{F}$. Then $f \sim g$ if $f_1 \sim g_1$, since we can reparametrize the functions k_1 and h_1 as well.
- It follows from the previous remark that it will be enough to prove the statements for functions $f, g \in \mathcal{F}$ instead of for all functions satisfying $f \bowtie g$.
- If $f, g \in \mathcal{F}$, then $f \sim g \iff \exists k, h \in \mathcal{F} : f \circ k = g \circ h$, which means that the new, more general definition of *"climbable"* is an extension of $(*)$.

First we shall prove a statement which is weaker than our theorem, but from which the statement of [2] follows easily.

Proposition 1. If $f \bowtie g$ and f is piecewise strictly monotone then $f \sim g$.

Proof. Let n denote the number of local extreme values of f. Since f has a global maximum and a global minimum, $n \geq 2$.

We shall prove the statement by induction on n. If $n = 2$, then f is strictly monotone, and in this case the statement is obvious.

Assume that $n \geq 3$ and the statement is valid for $2, ..., n-1$. By using the remarks above, we can suppose that $f, g \in \mathcal{F}$. First we assume that 1 is the only point where the value of f equals 1.

Let y_1 be the second largest local extreme value of f, let x_1 be the greatest point where the value of f is y_1. Let x_0 be the last local extremum point, let $y_0 = f(x_0)$. (See fig. 2.a.) (Here and the sequel *"last"* means the maximal element of a closed set.) It is clear that $x_0 < x_1$, $y_0 < y_1$, f is strictly increasing in $[x_0, 1]$, and $f([0, x_1]) = [0, y_1]$.

Let z_1 be the smallest value in $[0, 1]$ satisfying $g(z_1) = y_1$, u_1 is the smallest value in $[z_1, 1]$ satisfying $g(u_1) = y_0$, z_2 is the smallest value in $[u_1, 1]$ satisfying $g(z_2) = y_1$ ...etc. (See fig. 2.b.) The function g is countinuous, thus there exists an index m such that after z_m, $g > y_0$.

Let v_k be the greatest point in $[0, u_k]$ for which $g(v_k) = y_1$. (Clearly $z_k \leq v_k < u_k$.) Let w_k be a minimum point of g in $[v_k, z_{k+1}]$. (See fig. 2.b.) Let w'_k be the greatest point in $[0, x_1]$ for which $f(w'_k) = g(w_k)$.

Using these points and our assumption we can show how to *"climb"* f and g. It follows from the construction that

(a) $$f|[0, x_1] \bowtie g|[0, z_1]$$

(b) $$f|[x_1, w'_k] \bowtie g|[v_k, w_k]$$

(c) $$f|[w'_k, x_1] \bowtie g|[w_k, z_{k+1}],$$

and in these intervals f has less than n local extreme values, which means, using the assumption, that these pairs of parts of functions are climbable. We can link the climbing of the pairs of function parts (b) and (c). We still have to climb the function g in the intervals $[z_k, v_k]$ $(k = 1, 2, ..., m-1)$ in such a way that at the same time we traverse on the graph of f beginning from $(x_1, f(x_1))$ and returning there. Finally we also need that when we move on the graph of g from $(z_m, g(z_m))$ to $(1, 1)$, we similary move on the graph of f from $(x_1, f(x_1))$ to $(1, 1)$. We can do the above steps easily since in these intervals the range of g is in $[y_0, 1]$, $x_1 \in [x_0, 1]$ and f is strictly monotone increasing in $[x_0, 1]$, and its range is the interval $[y_0, 1]$ here and $g(1) = 1$. (For example $h(x) = x$ and $k = f^{-1} \circ g$ are well defined in the given intervals and obviously $f \circ k = g \circ h$.)

Therefore we can make the induction step in the case when 1 is the only point where the value of f is 1. Now we remove this assumption. The points where f attains its maximum are $x_1 < x_2 < ... < x_m = 1$, let y_k be the first minimum point of f in $[x_k, x_{k+1}]$, let z_k be the greatest point for which $g(z_k) = f(y_k)$. In this case $f|[0, x_1] \bowtie g$, $f|[x_k, y_k] \bowtie g|[1, z_k]$ and $f|[y_k, x_{k+1}] \bowtie g|[z_k, 1]$. For these functions we proved that they are climbable, we can link these climbings, so $f \sim g$. This concludes the proof.

Notation 2.

For a fixed $f \in \mathbf{C}[a, b]$ we shall say that $a = x_0, x_1, ..., x_n = b$ is a *nice sequence* if $f|[x_{i-1}, x_i] \in \mathcal{G}$ $(i = 1, 2, ..., n)$, in other words, if $f([x_{i-1}, x_i]) = [f(x_{i-1}), f(x_i)]$ and $x_{i-1} \neq x_i$.

We call a nice sequence a *nice partition* if $a = x_0 < x_1 < ... < x_n = b$.

A nice sequence or partition $x_0, x_1, ..., x_n$ is δ- *fine* if $|x_i - x_{i-1}| \leq \delta$ $(i = 1, ..., n)$.

A nice sequence or partition $x_0, ..., x_n$ (for f) is *oscillating* if

$$(f(x_i) - f(x_{i-1}))(f(x_{i+1}) - f(x_i)) < 0 \quad (i = 1, 2, ..., n-1).$$

Proposition 2. Suppose $f \in \mathbf{C}[a, b]$, f is not constant in any interval, and f is locally increasing or decreasing at a from the right and also at b from the left.

Then there is a nice oscillating partition for f.

Proof. Let $x_0 = a$.

Assume that we defined $x_0, ..., x_i$ (that is, $x_0, ..., x_i$ is a nice partition for $f|[a, x_i]$). Suppose that f is locally increasing or decreasing at x_i from the right. These assumptions hold for the first step.

Clearly we can assume that f is locally increasing at x_i from the right. We will distinguish between two cases.

Case A : The function f does not take the value of $f(x_i)$ in the interval $(x_i, b]$. (It follows that $f > f(x_i)$ here.)

In this case let x_{i+1} be the last global maximum point in $[x_i, b]$. If $x_{i+1} = b$, then the procedure is finished.

Case B : The function f does take the value of $f(x_i)$ in the interval $(x_i, b]$. Since f locally increases at x_i from the right, there exists a number x_i' in (x_i, b) for which $f(x_i') = f(x_i)$ and $f(x) \geq f(x_i)$ for every $x_i \leq x \leq x_i'$. Let x_{i+1} be the last global maximum point in $[x_i, x_i']$. In this way, $x_i < x_{i+1} < x_i'$.

In both cases x_{i+1} is a local maximum point, so if $x_{i+1} < b$, then f is locally decreasing at x_{i+1} from the right. On the other hand $f|[x_i, x_{i+1}] \in \mathcal{G}$, therefore the assumptions hold for the next step. We can continue the procedure, and we will have a nice partition if we reach the end. It is also clear that this will be a nice oscillating partition.

Therefore it will be suffecient to show that this procedure finishes after a finite number of steps. Let us suppose, to the contrary, that the procedure is infinite. Let x_n tend to the limit C. Again we will distinguish between two cases:

Case 1. There exists a step, say the i-th, when we used Case A. Since the point x_i we get in this step is the last maximum or the last minimum point in $[x_{i-1}, b]$, f never takes on the value of $f(x_i)$ in $(x_i, b]$, so at the next step Case A appears again, therefore continuing after this Case A will always occur.

We can not have $C = b$ since there is a left neighborhood of b where b is an extreme point, so if x_n is in this neighborhood then the procedure is finished in two steps.

Therefore $C < b$. Since f is continuous, $f(x_n) \to f(C)$, on the other hand the elements of the sequence $\{f(x_n)\}$ are alternating minimum and maximum values in an interval containing $[C, b]$, and this implies that $f|[C, b] \equiv f(C)$, but we assumed that f is not constant in any interval. Therefore this case can not happen.

Case 2. Only Case B occurs.
Let us suppose that f is locally increasing at x_i from the right. In this case $f(x_i) = f(x_i') < f(x_{i+1})$, x_{i+1} is the last maximum point in $[x_i, x_i']$, so in the next step $x_{i+1}' > x_i'$. x_{i+2} is a minimum point in $[x_{i+1}, x_{i+1}']$ and $x_i' \in [x_{i+1}, x_{i+1}']$, therefore $f(x_{i+2}) \leq f(x_i') = f(x_i)$. Continuing this argument we obtain $f(x_i) \geq f(x_{i+2}) \geq f(x_{i+4}) \geq ...$ By similar arguments $f(x_{i+1}) \leq f(x_{i+3}) \leq f(x_{i+5}) \leq ....$ On the other hand $f(x_i) < f(x_{i+1})$ so $\{f(x_n)\}$ cannot be convergent. But $f(x_n)$ tends to $f(C)$.

Therefore we get a contradiction in both cases, so the procedure is finite, and we proved previously that it gives a nice oscillating partition for f.

Proposition 3. Suppose that $\delta > 0$, $f \in \mathbf{C}[a, b]$, f is not constant in any interval, and f is locally increasing or decreasing at a from the right and also at b from the left.

a) Then there exists a nice δ-fine partition for f.

b) If we also suppose that f is not monotone in any interval then there exists a nice oscillating δ-fine partition, too.

Proof. Take an arbitrary $\delta/2$-fine partition $a = y_0 < ... < y_n = b$ of $[a, b]$. Let $z_0 = a$, $z_{n+1} = b$. For $i = 1, 2, ..., n$, if f is monotone in $[y_{i-1}, y_i]$, then let z_i be an arbitrary interior point of the interval, otherwise let z_i be a local extreme point in (y_{i-1}, y_i).

We can apply Proposition 2 for $f|[z_{i-1}, z_i]$. We can link the nice partitions, that we have so far, so we get a nice δ- fine partition for f. If f is not monotone in any interval then all the z_i $(i = 1, .., n)$ are local extreme points, for this reason, in this case, if we link the nice oscillating partitions the property of oscillation transfers, so we get a nice oscillating δ-fine partition.

The following two statements will construct two nice matching sequences for two matching functions.

Proposition 4. Suppose that $f \bowtie g$, f is not monotone in any interval and $\delta > 0$. Then there exists a nice δ- fine sequence $u_0, u_1, ..., u_m$ for f, and a nice sequence $v_0, ..., v_m$ for g, such that $f(u_i) = g(v_i) \quad (i = 0, ..., m)$.

Proof. Again we can assume that $f, g \in \mathcal{F}$. Applying Proposition 3.b to f, we know that there exists a nice oscillating δ-fine partition $0 = x_0 < x_1 < ... < x_n = 1$ for f. Let $f_1(x_i) = f(x_i) \quad (i = 0, 1, ..., n)$ and let f_1 be linear between x_i and $x_{i+1} \quad (i = 0, ..., n-1)$. We have $f_1 \bowtie g$ and f_1 is piecewise strictly monotone, therefore $f_1 \sim g$ by Proposition 1, that is there exist $k, h \in \mathcal{F}$ such that $f_1 \circ h = g \circ k$.

Look at the function h (See fig.3). Let α_1 be the smallest value in $[0, 1]$ satisfying $h(\alpha_1) = x_1$. Let α_2 be the smallest value in $[\alpha_1, 1]$ for which the value of h is x_0 or x_2, and generally if $h(\alpha_i) = x_j$, then α_{i+1} is the smallest value in $[\alpha_i, 1]$ for which the value of h is x_{j-1} or x_{j+1}. Since h is continuous, our process terminates after a finite number of steps, which means that there exists an index l such that $h(\alpha_l) = 1 (= x_n)$ and $h|[\alpha_l, 1] > x_{n-1}$.

Let γ_i be the greatest point in $[0, \alpha_{i+1}]$ satisfying $h(\gamma_i) = h(\alpha_i)$. Let $\gamma_0 = 0$, $\gamma_l = 1$. It is clear that $\alpha_{i+1} > \gamma_i \geq \alpha_i$ and $h|[\gamma_i, \alpha_{i+1}] \in \mathcal{G} \quad (i = 1, ..., l-1)$.

Let j be the index for which $h(\gamma_i) = h(\alpha_i) = x_j$. Since $g \circ k = f_1 \circ h$,

$$(1) \qquad g(k(\gamma_i)) = f_1(h(\gamma_i)) = f_1(x_j) = f(x_j) = f(h(\gamma_i))$$

$$(2) \qquad g(k(\alpha_i)) = f_1(h(\alpha_i)) = f_1(x_j) = f(h(\alpha_i)).$$

By definition $h(\alpha_{i+1}) = x_{j+\varepsilon}$ where $\varepsilon = \pm 1$, $h([\gamma_i, \alpha_{i+1}]) = [h(\gamma_i), h(\alpha_{i+1})] = [x_j, x_{j+\varepsilon}]$. On the other hand $x_0, ..., x_n$ is a nice partition for f, therefore

$$(3) \qquad f([h(\gamma_i), h(\alpha_{i+1})]) = f([x_j, x_{j+\varepsilon}]) = [f(x_j), f(x_{j+\varepsilon})] = [f(h(\gamma_i)), f(h(\alpha_{i+1}))]$$

and

$$g([k(\gamma_i), k(\alpha_{i+1})]) \subset g(k([\gamma_i, \alpha_{i+1}])) = f_1(h([\gamma_i, \alpha_{i+1}])) =$$

$$= f_1([h(\gamma_i), h(\alpha_{i+1})]) = [f_1(h(\gamma_i)), f_1(h(\alpha_{i+1}))] = [g(k(\gamma_i)), g(k(\alpha_{i+1}))].$$

Therefore obviously

$$g([k(\gamma_i), k(\alpha_{i+1})]) = [g(k(\gamma_i)), g(k(\alpha_{i+1}))]. \tag{4}$$

(3) and (4) imply that

$$f|[h(\gamma_i), h(\alpha_{i+1})] \in \mathcal{G} \text{ and } g|[k(\gamma_i), k(\alpha_{i+1})] \in \mathcal{G} \quad (i = 0, ..., l-1). \tag{5}$$

Since $x_0, ..., x_n$ is a nice oscillating partition for f, $h(\alpha_i) = x_j$ is a local extremum of f. We can assume that this is a local minimum point. In this case $f(x_{j-1}) > f(x_j) < f(x_{j+1})$. Let β_i be a maximum point of $f_1 \circ h = g \circ k$ in $[\alpha_i, \gamma_i]$, let M_i be the maximum value here. From the definition, $h([\alpha_i, \gamma_i]) \subset [x_{j-1}, x_{j+1}]$, $f_1([x_{j-1}, x_{j+1}]) \geq f(x_j)$, therefore for all $t \in [\alpha_i, \gamma_i]$ we have

$$g(k(\beta_i)) = M_i \geq g(k(t)) = f_1(h(t)) \geq f(x_j) = g(k(\alpha_i)) = g(k(\gamma_i)).$$

Therefore

$$g(k([\alpha_i, \gamma_i])) \subset [f(x_j), M_i] = [g(k(\alpha_i)), g(k(\beta_i))].$$

On the other hand

$$g(k([\alpha_i, \gamma_i])) \supset g(k([\alpha_i, \beta_i])) \supset g([k(\alpha_i), k(\beta_i)]).$$

So we obtain $g([k(\alpha_i), k(\beta_i)]) = [g(k(\alpha_i)), g(k(\beta_i))]$. By simular arguments

$$g([k(\beta_i), k(\gamma_i)]) = [g(k(\beta_i)), g(k(\gamma_i))],$$

therefore

$$g|[k(\beta_i), k(\gamma_i)], g|[k(\alpha_i), k(\beta_i)] \in \mathcal{G} \tag{6}$$

if $k(\beta_i) \neq k(\gamma_i)$, $k(\alpha_i) \neq k(\beta_i)$, which holds if and only if $\alpha_i \neq \gamma_i$.

Now $h([\alpha_i,\gamma_i]) \subset [x_{j-1},x_{j+1}]$ implies $f_1 \circ h([\alpha_i,\gamma_i]) \subset f_1([x_{j-1},x_{j+1}]) = f([x_{j-1},x_{j+1}])$. Therefore f takes the value of M_i in $[x_{j-1},x_{j+1}]$. Let z_i be the nearest such point to x_j. In this case $f([x_j,z_i]) = [f(x_j),\beta_i] = [f(x_j),f(z_i)]$. Therefore

$$f|[h(\alpha_i),z_i] = f|[z_i,h(\gamma_i)] \in \mathcal{G} \text{ if } \alpha_i \neq \gamma_i \quad (i=1,...,l) \tag{7}$$

$$g(k(\beta_i)) = M_i = f(z_i) \quad (i=1,...,l). \tag{8}$$

Let $\{u_i\}$ and $\{v_i\}$ be the sequences obtained from the sequences

$$0 = h(\gamma_0), h(\alpha_1), z_1, h(\gamma_1), h(\alpha_2), z_2, h(\gamma_2), ..., h(\alpha_l), z_l, h(\gamma_l) = 1$$

and

$$0 = k(\gamma_0), k(\alpha_1), k(\beta_1), k(\gamma_1), k(\alpha_2), k(\beta_2), k(\gamma_2), ..., k(\gamma_l), k(\beta_l), k(\gamma_l) = 1$$

after omitting the repeating terms. (We have to omit z_i, $h(\gamma_i)$, $k(\beta_i)$ and $k(\gamma_i)$ when $\alpha_i = \gamma_i$.)

(5),(6) and (7) show that these are nice sequences for f and g. (1),(2) and (8) show that $f(u_i) = g(v_i)$ $(i = 0,...,m)$. $x_0 < ... < x_n$ is δ-fine and $z_i \in [x_{j-1},x_{j+1}]$, so $\{u_i\}$ is δ-fine. Therefore we have the desired sequences.

The following statement is an almost trivial corollary of the preceding statement:

Proposition 5. Suppose $f \bowtie g$ with neither f nor g monotone in any interval and $\delta > 0$.

Then there exists a nice sequence for f and a nice sequence for g such that both sequences are δ-fine and $f(x_i) = g(y_i)$ $(i = 0,...,n)$.

Proof. We can apply Proposition 4 to f, g and δ, so we get the sequences $\{u_i\}$ and $\{v_i\}$. Since $f(u_i) = g(v_i)$, $f|[u_i,u_{i+1}] \bowtie g|[v_i,v_{i+1}]$ $(i = 0,...,m-1)$. We can apply Proposition 4 to $g|[v_i,v_{i+1}]$, $f|[u_i,u_{i+1}]$ and δ instead of f, g and δ. Linking the constructed sequences, we will obviously get the desired sequences $\{x_i\}$ and $\{y_i\}$.

Using this statement we can prove that any pair of functions in a large class of functions is climbable.

Proposition 6. If $f, g \in \mathcal{F}$ and neither f nor g is monotone in any interval then $f \sim g$.

Proof. Let $\delta_1 = 1/2$. Let us use the preceding statement for f, g and δ_1, we get the nice δ_1-fine sequences $\{x_i\}$ and $\{y_i\}$. For every $i \in \{0, 1, ..., n\}$, let $h_1(i/n) = x_i$, $k_1(i/n) = y_i$, and let h_1 and k_1 be linear between two such points.

Let $\delta_2 < \min\{1/2^2, |x_i - x_{i-1}|, |y_i - y_{i-1}| : i = 1, ..., n\}$. Applying Proposition 5 to the functions $f|[x_{i-1}, x_i] \bowtie g|[y_{i-1}, y_i]$ and δ_2 we get the nice sequences $\{x_{i,j}\}_{j=0}^{n_i}$ and $\{y_{i,j}\}_{j=0}^{n_i}$. Let

$$h_2(\frac{i}{n} + \frac{j}{n \cdot n_1}) = x_{i,j},$$
$$k_2(\frac{i}{n} + \frac{j}{n \cdot n_1}) = y_{i,j},$$

and let h_2 and k_2 be linear between two such points.

Let $\delta_3 < \min\{1/2^3, |x_{i,j} - x_{i,j-1}|, |y_{i,j} - y_{i,j-1}| : i \in \{1, ..., n\},\ j \in \{1, ..., n_i\}\}$, and continue this procedure infinetely. We get the functions $h_1, h_2, h_3, ...$ and $k_1, k_2, k_3, ...$ in $\mathcal{F}$.

If $l_1 < l_2$, then $|h_{l_1} - h_{l_2}|, |k_{l_1} - k_{l_2}| < \delta_{l_1} < 1/2^{l_1}$, therefore $\{h_l\}$ and $\{k_l\}$ are uniformly convergent. Denote their limits by h and k, then $h, k \in \mathcal{F}$.

At the point

$$t = \frac{i_1}{n} + \frac{i_2}{n \cdot n_{i_1}} + ... + \frac{i_m}{n \cdot ... \cdot n_{i_1, ..., i_{m-1}}}$$

we have $h_m(t) = h_{m+1}(t) = h_{m+2}(t) = ...$, so $h(t) = h_m(t)$ and clearly $k(t) = k_m(t)$, too. Therefore

$$f \circ h(t) = f \circ h_m(t) = f(x_{i_1, ..., i_m}) = g(y_{i_1, ..., i_m}) = g \circ k_m(t) = g \circ k(t).$$

The number δ_m was chosen so small that for every $(i_1, ..., i_{m-1})$ we have $n_{i_1, ..., i_{m-1}} > 1$. For this reason the set of points

$$\frac{i_1}{n} + ... + \frac{i_m}{n \cdot ... \cdot n_{i_1, ..., i_{m-1}}}$$

is a dense set in $[0,1]$. Therefore the continuous functions $f \circ h$ and $g \circ k$ are equal in a dense set,which means $f \circ h = g \circ k$, in other words $f \sim g$.

After this it will be easy to prove the promised theorem.

Theorem. If $f, g \in \mathcal{F}$ and neither f nor g is constant in any interval then $f \sim g$.

Proof. Let us modify f in every maximal monotone portion, $[a,b] \subset [0,1]$, as follows. We replace f in $[a,b]$ by a function which is not monotone in any subinterval of $[a,b]$, equal to f at the points a and b, and its range is the interval $[f(a), f(b)]$. Denote the function we obtained by f_1. Put $f_2 = f^{-1} \circ f_1$ in the (strictly) monotone portions of f, and let f_2 be the identity function otherwise. This definition makes sense, $f_1 = f \circ f_2$, $f_1, f_2 \in \mathcal{F}$, and f_1 is not monotone in any interval.

We can do the same with g, and we get the functions g_1 and g_2. According to Proposition 6, there exists h_1 and k_1 in $\mathcal{F}$ such that $f_1 \circ h_1 = g_1 \circ k_1$.

Let $h = f_2 \circ h_1$, $k = g_2 \circ k_1$. Then $h, k \in \mathcal{F}$ and

$$f \circ h = f \circ f_2 \circ h_1 = f_1 \circ h_1 = g_1 \circ k_1 = g \circ g_2 \circ k_1 = g \circ k,$$

which means that $f \sim g$.

Remarks. This theorem (in fact also Proposition 6) shows that two typical functions from $\mathcal{F}$ are climbable. Of course, it would be interesting to characterize nonclimbable pairs (f,g). Our conjecture is that there is no simple characterisation, maybe the set $\{(f,g) | f, g \in \mathcal{F}, f \not\sim g\}$ is not even a Borel subset of $\mathcal{F} \times \mathcal{F}$.

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