Periodic decomposition of measurable integer valued functions

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Abstract

We study those functions that can be written as a sum of (almost everywhere) integer valued periodic measurable functions with given periods. We show that being (almost everywhere) integer valued measurable function and having a real valued periodic decomposition with the given periods is not enough. We characterize those periods for which this condition is enough.

We also get that the class of bounded measurable (almost everywhere) integer valued functions does not have the so called decomposition property. We characterize those periods $a_1, \dots, a_k$ for which an almost everywhere integer valued bounded measurable function f has an almost everywhere integer valued bounded measurable $(a_1, \dots, a_k)$-periodic decomposition if and only if $\Delta_{a_1} \dots \Delta_{a_k} f = 0$, where $\Delta_a f(x) = f(x+a) - f(x)$.

Introduction

In [8] those functions were studied that can be written as a sum of periodic integer valued functions with given periods $a_1, \dots, a_k$. Clearly these functions must be integer valued and they can be written as a sum of periodic real valued functions with given periods $a_1, \dots, a_k$. Several results were proved about the question whether the converse is true or false; that is, whether the existence of a real valued $(a_1, \dots, a_k)$-periodic decomposition of an integer valued function implies the existence of an integer valued $(a_1, \dots, a_k)$-periodic decomposition. Among others the following result were proved:

Theorem 0.1 *([8, Theorem 2.1]) Suppose that an integer valued function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = g_1 + \dots + g_k$, where each g_j is a real valued a_j-periodic function for some $a_j \in \mathbb{Z}$.*

Then f can be also written as $f = h_1 + \dots + h_k$, where each h_j is an integer valued a_j-periodic function.

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The question whether the same result holds for functions defined on $\mathbb{R}$ is still open:

Question 0.2 *[8, Question 5.1] Is it true for any $a_1, \ldots, a_k \in \mathbb{R}$ that if an integer valued function $f : \mathbb{R} \to \mathbb{Z}$ has a real valued $(a_1, \ldots, a_k)$-periodic decomposition then f also has an integer valued $(a_1, \ldots, a_k)$-periodic decomposition?*

There are some positive partial results if we have some assumptions about the periods $a_1, \ldots, a_k$ (see [8] and [5]).

For bounded decomposition of bounded functions the following counter-example was found for the analogue question:

Theorem 0.3 *([8, Theorem 3.1]) There exists a function $u : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ that can be written as a sum of a $(0, 1)$-periodic, a $(1, 0)$-periodic and a $(1, 1)$-periodic bounded $\mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ function, can be written also as the sum of three periodic $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ functions with the same periods but cannot be written as a sum of three periodic bounded $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ functions with the same periods.*

Note that by repeating this construction on each coset one can get a similar counter-example on any Abelian group that contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup, so in particular there exist $a_1, a_2, a_3 \in \mathbb{R}$ and a function $f : \mathbb{R} \to \{0, 1\}$ that has a bounded real valued (a_1, a_2, a_3)-periodic decomposition but does not have a bounded integer valued (a_1, a_2, a_3)-periodic decomposition.

In the above cited results no measurability was assumed. In this paper we will study what happens if we allow only measurable functions. First we give a negative answer (Theorem 1.2) to the measurable analogue of Question 0.2 and then we characterize (Theorem 2.5) those periods for which we have positive result, at least if we are happy with almost everywhere integer valued decompositions. Everywhere integer valued measurable decompositions are studied in Section 3. It turns out that the question whether we can get integer valued decompositions instead of almost everywhere integer valued decompositions depends on the answers to the nonmeasurable questions like the above mentioned Question 0.2.

The characterization of those functions that can be written as a sum of periodic functions with given period has a much longer history. It started in the seventies with some unpublished work of I. Z. Ruzsa and continued among others in [1], [3], [5], [6], [7], [9], [10], [11], [12], [13], [14] and [15]. If $f = f_1 + \ldots + f_n$ is an $(a_1, \ldots, a_n)$-periodic decomposition of f then

$$\Delta_{a_1} \Delta_{a_2} \cdots \Delta_{a_n} f = 0, \quad \text{where} \quad \Delta_{a_j} f(x) = f(x + a_j) - f(x), \tag{1}$$

since the difference operators Δ_{a_j} commute. A class of functions $\mathcal{F}$ is said to have the *decomposition property* if every $f \in \mathcal{F}$ that satisfies (1) has an $(a_1, \ldots, a_k)$-periodic decomposition in $\mathcal{F}$. Since for the identity function $f(x) = x$ we clearly have $\Delta_1 \Delta_1 f = 0$ but it is not the sum of two 1-periodic functions, many natural classes of functions (e.g. all $\mathbb{R} \to \mathbb{R}$ functions, continuous $\mathbb{R} \to \mathbb{R}$ functions) do not have the decomposition property. However, many classes of functions do have the decomposition property: for example the class of all bounded continuous $\mathbb{R} \to \mathbb{R}$ functions [11], the class of all bounded $\mathbb{R} \to \mathbb{R}$ functions [3], [12], the class of all bounded measurable $\mathbb{R} \to \mathbb{R}$ functions [12] and the class of all bounded real valued functions on an arbitrary Abelian group [12].

For integer valued functions it was proved in [8] that the class of bounded $\mathbb{Z} \to \mathbb{Z}$ functions has the decomposition property but the class of bounded $\mathbb{R} \to \mathbb{Z}$ functions does not have the decomposition property. In fact, among the torsion free Abelian groups only the additive subgroups of $\mathbb{Q}$ are those on which the class of bounded integer valued functions has the decomposition property [8, Corollary 3.5].

In this note we get (Corollary 1.3) that on $\mathbb{R}$ the classes of bounded measurable (almost everywhere) integer valued functions and (almost everywhere) integer valued L_∞ functions do not have the decomposition property. We characterize (Theorem 2.5) those periods $a_1, \ldots, a_k$ for which for any bounded measurable $\mathbb{R} \to \mathbb{Z}$ function the existence of a bounded measurable almost everywhere integer valued $(a_1, \ldots, a_k)$-periodic decomposition is equivalent to (1). We show (Proposition 3.4) that this characterization is not valid for everywhere integer valued $(a_1, \ldots, a_k)$-periodic decompositions and finally we give a conjecture about a possible characterization of those periods $a_1, \ldots, a_k$ for which a bounded measurable integer valued function has a bounded measurable integer valued $(a_1, \ldots, a_k)$-periodic decomposition if and only if (1) holds.

Meanwhile, as a spin-off, we also characterize (Corollary 2.4) those periods $a_1, \ldots, a_k$ for which the measurable $(a_1, \ldots, a_k)$-periodic decomposition of an $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ function is essentially unique. This characterization turns out to be different for $\mathbb{R} \to \mathbb{R}$ functions (Lemma 1.1).

1 A negative result

The following fact is known, see e.g. in [9].

Lemma 1.1 *Let $a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\}$ such that $a_i/a_j \notin \mathbb{Q}$ for any $i \neq j$ and suppose that $f_1 + \ldots + f_k = g_1 + \ldots + g_k$ and for each j, f_j and g_j are a_j-periodic measurable $\mathbb{R} \to \mathbb{R}$ functions.*

Then $f_j - g_j$ is almost everywhere constant for every $j = 1, \ldots, k$.

It is easy to see that the condition $a_i/a_j \notin \mathbb{Q}$ for $i \neq j$ is also necessary. We will see (Corollary 2.4) that for $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ functions the necessary and sufficient condition for $a_1, \ldots, a_k$ is stronger.

The following theorem shows that the existence of a measurable real valued $(a_1, \ldots, a_n)$-periodic decomposition for an $\mathbb{R} \to \mathbb{Z}$ function does not always implies the existence of a measurable integer valued $(a_1, \ldots, a_n)$-periodic decomposition, not even the existence of a measurable, almost everywhere integer valued $(a_1, \ldots, a_n)$-periodic decomposition.

Theorem 1.2 *There exists an integer valued bounded Lebesgue measurable function on the real line that can be written as a sum of three real valued bounded measurable periodic functions but cannot be written as a sum of three almost everywhere integer valued measurable periodic functions with the same periods.*

Proof. Let $t \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary and let

$$f(x) = \{tx\} + \{(1-t)x\} + \{-x\}, \tag{2}$$

where $\{.\}$ denotes the fractional part; that is, $\{a\} = a - [a]$. Then f is clearly measurable and it is the sum of a $\frac{1}{t}$-periodic a $\frac{1}{1-t}$-periodic and a 1-periodic bounded measurable function. Noting that f can be also written as

$$f(x) = tx - [tx] + (1-t)x - [(1-t)x] - x - [-x] = -[tx] - [(1-t)x] - [-x]$$

we get that f is integer valued.

Suppose that $f = g_1 + g_2 + g_3$ and g_1, g_2, g_3 are measurable almost everywhere integer valued periodic measurable functions with periods $\frac{1}{t}$, $\frac{1}{1-t}$ and 1, resp. Since $t \notin \mathbb{Q}$ and by adding a constant to $\{-x\}$ we cannot get an almost everywhere integer valued function, applying Lemma 1.1 for $\{tx\} + \{(1-t)x\} + \{-x\} = g_1 + g_2 + g_3$, we get a contradiction. $\square$

Corollary 1.3 *The following classes of functions do not have the decomposition property:* $\{f : \mathbb{R} \to \mathbb{Z} \mid f \in L_\infty\}$, $\{f : \mathbb{R} \to \mathbb{Z} \mid f$ *is bounded and measurable*$\}$, $\{f : \mathbb{R} \to \mathbb{R} \mid f \in L_\infty$ *and* f *is almost everywhere integer valued*$\}$, *and* $\{f : \mathbb{R} \to \mathbb{R} \mid f$ *is bounded, measurable and almost everywhere integer valued*$\}$.

Proof. Let t and f be as in the above proof. Then f is contained in all the above classes, $\Delta_1 \Delta_{1/t} \Delta_{1/(1-t)} f = 0$ but, as we saw it in the above proof, f cannot be written as a sum of a $\frac{1}{1-t}$-periodic, a $\frac{1}{t}$ and a 1-periodic measurable almost everywhere integer valued function. $\square$

2 Almost everywhere integer valued decompositions

The following lemma might be known but for completeness we present a proof.

Lemma 2.1 *If* $E_j \subset \mathbb{R}$ *is an* a_j*-periodic measurable set with positive measure for each* $j = 1, \ldots, k$ *and* $\frac{1}{a_1}, \ldots, \frac{1}{a_k}$ *are linearly independent over* $\mathbb{Q}$ *then* $E_1 \cap \ldots \cap E_k$ *has positive measure.*

Proof. By applying the Lebesgue's Density Theorem for each E_j, we can find $\delta > 0$ and $d_1, \ldots, d_k \in \mathbb{R}$ such that

$$\lambda((d_j - 2\delta, d_j + 2\delta) \setminus E_j) < \frac{2\delta}{k} \qquad (j = 1, \ldots, k).$$

For each $j = 1, \ldots, k$, using that E_j is a_j-periodic, we get that for any $m_j \in \mathbb{Z}$,

$$t \in (m_j a_j + d_j - \delta, m_j a_j + d_j + \delta) \Longrightarrow \lambda\left((t - \delta, t + \delta) \setminus E_j\right) < \frac{2\delta}{k}. \qquad (3)$$

One form of Kronecker's theorem (see e.g. in [4, Theorem 444]) states that if $b_1, \ldots, b_k \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, $c_1, \ldots, c_k \in \mathbb{R}$ and $\varepsilon > 0$ then there exists $t \in \mathbb{R}$ and $m_1, \ldots, m_k \in \mathbb{Z}$ such that $|b_j t - m_j - c_j| < \varepsilon$ for every $j = 1, \ldots, k$.

Applying the above mentioned form of Kronecker's theorem for $b_j = \frac{1}{a_j}, c_j = \frac{d_j}{a_j}$ and $\varepsilon = \frac{\delta}{a_j}$ we get a $t \in \mathbb{R}$ such that $|t - m_j a_j - d_j| < \delta$ for every j.

Then by (3),

$$\lambda\left((t - \delta, t + \delta) \setminus E_j\right) < \frac{2\delta}{k}$$

for every $j = 1, \ldots, k$, which implies that $\lambda(E_1 \cap \ldots \cap E_k \cap (t-\delta, t+\delta)) > 0$. $\square$

We remark that Lemma 2.1 easily implies the following statement. In fact, the converse implication is also easy.

Corollary 2.2 *If $f_1, \ldots, f_k : \mathbb{R} \to (0, \infty)$ are periodic measurable functions such that the reciprocals of the periods are linearly independent over $\mathbb{Q}$ then*

$$||f_1 + \ldots + f_k||_\infty = ||f_1||_\infty + \ldots + ||f_k||_\infty.$$

$\square$

The following theorem shows that if $\frac{1}{a_1}, \ldots, \frac{1}{a_k}$ are linearly independent over $\mathbb{Q}$ then the almost everywhere integer valued measurable functions have only trivial measurable $(a_1, \ldots, a_k)$-periodic decompositions.

Theorem 2.3 *Let $a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\}$ such that $\frac{1}{a_1}, \ldots, \frac{1}{a_k}$ are linearly independent over $\mathbb{Q}$. Suppose that $f_j : \mathbb{R} \to \mathbb{R}$ is an a_j-periodic measurable function for each $j = 1, \ldots, k$ and that $f = f_1 + \ldots + f_k$ is an almost everywhere integer valued function.*

Then each fractional part $\{f_j\}$ is constant almost everywhere.

Proof. Let

$$F_j = \bigcup \left\{ f_j^{-1}((r,q)) \ : \ r, q \in \mathbb{Q}, \ \lambda\big(f_j^{-1}((r,q))\big) = 0 \right\} \qquad (j = 1, \ldots, k),$$

$$E = \{x \in \mathbb{R} \ : f(x) \in \mathbb{Z}\} \setminus \bigcup_{j=1}^{k} F_j.$$

Then $\lambda(\mathbb{R} \setminus E) = 0$, so it enough to prove that for any fixed $u, v \in E$ and $\varepsilon > 0$ we have $||f_1(u) - f_1(v)|| < \varepsilon$, where $||.||$ denotes the distance from the nearest integer; that is, $||x|| = \min(\{x\}, \{1-x\})$.

Let

$$E_1 = \left\{ x \in \mathbb{R} \ : \ |f_1(x) - f_1(u)| < \frac{\varepsilon}{k} \right\}$$

and

$$E_j = \left\{ x \in \mathbb{R} \ : \ |f_j(x) - f_j(v)| < \frac{\varepsilon}{k} \right\} \qquad (j = 2, 3, \ldots, k).$$

For each $j = 1, \ldots, k$ the set E_j is measurable and a_j-periodic since f_j is measurable and a_j-periodic, and $\lambda(E_j) > 0$ since $u, v \in E$ and so $u, v \notin F_j$. Hence by Lemma 2.1, $\lambda(E_1 \cap \ldots \cap E_k) > 0$, so there exists a $t \in E \cap E_1 \cap \ldots \cap E_k$.

Then

$$|f_1(u) - f_1(t)| < \frac{\varepsilon}{k}, \tag{4}$$

$$|f_j(v) - f_j(t)| < \frac{\varepsilon}{k} \quad (j = 2, \ldots, k). \tag{5}$$

Since $f_1(v) + f_2(v) + \ldots + f_k(v) = f(v) \in \mathbb{Z}$ and $f_1(t) + f_2(t) + \ldots + f_k(t) = f(t) \in \mathbb{Z}$, (5) implies that $||f_1(t) - f_1(v)|| < (k-1)\frac{\varepsilon}{k}$. Combining this with (4) we get that $||f_1(u) - f_1(v)|| < \varepsilon$, which completes the proof. $\square$

Now we can characterize those periods for which the measurable decomposition of an $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ function is unique up to additive constants. Note that, by Lemma 1.1, the characterization is different for $\mathbb{R} \to \mathbb{R}$ functions.

Corollary 2.4 *For any $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$ the following two statements are equivalent.*

(i) If $f_1 + \ldots + f_k = g_1 + \ldots + g_k$ and for each j, f_j and g_j are a_j-periodic measurable $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ functions then $f_j - g_j$ is almost everywhere constant for every $j = 1, \dots, k$.

(ii) $\frac{1}{a_1}, \dots, \frac{1}{a_k}$ are linearly independent over $\mathbb{Q}$.

Proof. $(i) \Rightarrow (ii)$: Suppose that (ii) is false, so there exists $(m_1, \dots, m_k) \in \mathbb{Z} \times \ldots \times \mathbb{Z} \setminus \{0, \dots, 0\}$ such that

$$\frac{m_1}{a_1} + \ldots + \frac{m_k}{a_k} = 0.$$

Then $f_j(x) = \frac{m_j}{a_j} x \mod 1$ for $j = 1, \dots, k$ and $g_1 = \ldots = g_k = 0$ shows that (i) is also false.

$(ii) \Rightarrow (i)$: This follows simply from Theorem 2.3 □

Now we can characterize those periods for which the existence of a (bounded) measurable real valued $(a_1, \dots, a_k)$-periodic decomposition of an integer valued or almost everywhere integer valued function implies the existence of a (bounded) measurable almost everywhere integer valued $(a_1, \dots, a_k)$-periodic decomposition. We will get the same characterization for those periods for which an integer valued or almost everywhere integer valued bounded measurable function has a bounded measurable almost everywhere integer valued decomposition if and only if $\Delta_{a_1} \ldots \Delta_{a_k} f = 0$, where $\Delta_a f(x) = f(x+a) - f(x)$.

Theorem 2.5 *For any $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$ the following seven statements are equivalent.*

$(i)/(i')$ If an everywhere/almost everywhere integer valued measurable function f on $\mathbb{R}$ can be decomposed as $f = f_1 + \ldots + f_k$ such that each f_j is an a_j-periodic measurable $\mathbb{R} \to \mathbb{R}$ function then f can be also decomposed as $f = g_1 + \ldots + g_k$ such that each g_j is an a_j-periodic almost everywhere integer valued measurable function.

$(ii)/(ii')$ If an everywhere/almost everywhere integer valued bounded measurable function f on $\mathbb{R}$ can be decomposed as $f = f_1 + \ldots + f_k$ such that each f_j is an a_j-periodic bounded measurable $\mathbb{R} \to \mathbb{R}$ function then f can be also decomposed as $f = g_1 + \ldots + g_k$ such that each g_j is an a_j-periodic almost everywhere integer valued bounded measurable function.

$(iii)/(iii')$ An everywhere/almost everywhere integer valued bounded measurable function f on $\mathbb{R}$ can be decomposed as $f = g_1 + \ldots + g_k$ such that each g_j is an a_j-periodic almost everywhere integer valued bounded measurable function if and only if $\Delta_{a_1} \ldots \Delta_{a_k} f = 0$.

(iv) If $B_1, \dots, B_n$ are the equivalence classes of $\{a_1, \dots, a_k\}$ with respect to the relation $a \sim b \Leftrightarrow a/b \in \mathbb{Q}$, and b_j denotes the smallest common multiple of the numbers in B_j (for each $j = 1, \dots, n$) then $\frac{1}{b_1}, \dots, \frac{1}{b_n}$ are linearly independent over $\mathbb{Q}$.

Proof. $(i) \Leftrightarrow (i')$, $(ii) \Leftrightarrow (ii')$, $(iii) \Leftrightarrow (iii')$: The $\Leftarrow$ implications are clear. Now we prove $(i) \Rightarrow (i')$, the other $\Rightarrow$ implications can be proved the same way.

Suppose that $f : \mathbb{R} \to \mathbb{R}$, $H \subset \mathbb{R}$ has measure zero, f is integer valued on $\mathbb{R} \setminus H$ and $f = f_1 + \ldots + f_k$ is a real valued $(a_1, \ldots, a_k)$-periodic decomposition of f. We need to find an almost everywhere integer valued $(a_1, \ldots, a_k)$-periodic decomposition of f.

By replacing H by $\{h+n_1a_1+\ldots+n_ka_k : h \in H, n_1, \ldots, n_k \in \mathbb{Z}\}$ if necessary we can suppose that the characteristic functions χ_H and $\chi_{\mathbb{R}\setminus H}$ are a_j-periodic for every $j = 1, \ldots, k$.

Applying (i) to the integer valued function $\chi_{\mathbb{R}\setminus H} \cdot f$ we get an almost everywhere integer valued $(a_1, \ldots, a_k)$-periodic decomposition $\chi_{\mathbb{R}\setminus H} \cdot f = g_1 + \ldots + g_k$. Multiplying by $\chi_{\mathbb{R}\setminus H}$ we get

$$\chi_{\mathbb{R}\setminus H} \cdot f = \chi_{\mathbb{R}\setminus H} \cdot g_1 + \ldots + \chi_{\mathbb{R}\setminus H} \cdot g_k. \tag{6}$$

By adding $\chi_H \cdot f = \chi_H \cdot f_1 + \ldots \chi_H \cdot f_k$ to (6) we get an almost everywhere integer valued $(a_1, \ldots, a_k)$-periodic decomposition

$$f = (\chi_{\mathbb{R}\setminus H} \cdot g_1 + \chi_H \cdot f_1) + \ldots + (\chi_{\mathbb{R}\setminus H} \cdot g_k + \chi_H \cdot f_k).$$

$(i) \Rightarrow (iv)$, $(ii) \Rightarrow (iv)$: We prove that if (iv) is false then (i) and (ii) are also false. Suppose that $\frac{1}{b_1}, \ldots, \frac{1}{b_n}$ are not linearly independent over $\mathbb{Q}$. For each b_j choose an a_{i_j} such that b_j is a multiple of a_{i_j}. Then $\frac{1}{a_{i_1}}, \ldots, \frac{1}{a_{i_n}}$ are also linearly dependent over $\mathbb{Q}$, so there exists $(m_1, \ldots, m_n) \in \mathbb{Z} \times \ldots \times \mathbb{Z} \setminus \{0, \ldots, 0\}$ such that

$$\frac{m_1}{a_{i_1}} + \ldots + \frac{m_n}{a_{i_n}} = 0. \tag{7}$$

Let

$$f_{i_j}(x) = \left\{ \frac{m_j}{a_{i_j}} x \right\} \qquad (j = 1, \ldots, n),$$

$$f_i(x) = 0 \qquad (i \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_n\})$$

and $f = f_1 + \ldots + f_k$.

Then, using (7), we get

$$f(x) = \sum_{i=1}^{k} f_i = \sum_{j=1}^{n} f_{i_j} = \sum_{j=1}^{n} \frac{m_j}{a_{i_j}} x - \sum_{j=1}^{n} \left[\frac{m_j}{a_{i_j}} x \right] = - \sum_{j=1}^{n} \left[\frac{m_j}{a_{i_j}} x \right] \in \mathbb{Z}.$$

Clearly each f_i is a bounded measurable a_i-periodic $\mathbb{R} \to \mathbb{R}$ function, so the conditions of (i) and (ii) are satisfied.

Suppose that f can be also written as $f = g_1 + \ldots + g_k$ such that each g_i is an a_i-periodic measurable almost everywhere integer valued function. For each $j = 1, \ldots, n$ let h_j be the sum of those g_i-s for which b_j is a multiple of a_i. Then $f = h_1 + \ldots + h_n$ and each h_j is an almost everywhere integer valued measurable b_j-periodic function. On the other hand $f = f_{i_1} + \ldots + f_{i_n}$ and each f_{i_j} is a measurable b_j-periodic function. Since $b_j / b_{j'} \notin \mathbb{Q}$ for any $j \neq j'$, Lemma 1.1 implies that $f_{i_j} - h_j$ is constant almost everywhere. Since $f_{i_j} = \{\frac{m_j}{a_{i_j}} x\}$, h_j is almost everywhere integer valued and at least one of $m_1, \ldots, m_n$ is not zero, this is a contradiction.

$(ii) \Leftrightarrow (iii)$: As we already mentioned in the Introduction, it is proved in [12] that the class of bounded measurable functions has the decomposition property; that is, a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ can be decomposed as $f = f_1 + \ldots + f_k$ such that each f_j is an a_j-periodic real valued bounded measurable function if and only if $\Delta_{a_1} \ldots \Delta_{a_k} f = 0$. On the other hand, by (ii), for integer valued functions the existence of a real valued bounded measurable $(a_1, \ldots, a_k)$-periodic decomposition is equivalent with the existence of an almost everywhere integer valued bounded measurable $(a_1, \ldots, a_k)$-periodic decomposition.

$(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$: First consider the case when $a_i/a_j \in \mathbb{Q}$ for every i, j. Then we can clearly assume that $a_j \in \mathbb{Z}$ for every j.

For $t \in [0,1)$ and $n \in \mathbb{Z}$ let $F_t(n) = f(n+t)$ and $F_{j,t}(n) = f_j(n+t)$ $(j = 1, \ldots, k)$. Then, applying Theorem 0.1 for each $t \in [0,1)$ for the decomposition $F_t = F_{1,t} + \ldots + F_{k,t}$ we get functions $G_{j,t} : \mathbb{Z} \to \mathbb{Z}$ such that $F_t = G_{1,t} + \ldots + G_{k,t}$ and each $G_{j,t}$ is a_j-periodic. Letting $g_j(n+t) = G_{j,t}(n)$ for each $j = 1, \ldots, k$, $n \in \mathbb{Z}$, $t \in [0,1)$ we get that $g_1, \ldots, g_k$ have all the desired properties except measurability and boundedness.

Let N be the smallest common multiple of $a_1, \ldots, a_k$. For every $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}$ let

$$A_{n_0,\ldots,n_{N-1}} = \{x \in [0,1) \ : f(x) = n_0, f(x+1) = n_1, \ldots, f(x+N-1) = n_{N-1}\}.$$

Note that if $t, t' \in A_{n_0,\ldots,n_{N-1}}$ for some $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}$ then $F_t = F_{t'}$, so we may guarantee $G_{j,t} = G_{j,t'}$ for each j in this case. Since every $A_{n_0,\ldots,n_{N-1}}$ is measurable this guarantees that $g_1, \ldots, g_k$ are measurable.

If $f_1, \ldots, f_k$ are bounded, then f is also bounded, so $A_{n_0,\ldots,n_{N-1}}$ is nonempty only for finitely many sequences $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}$. Since each $G_{j,t}$ is clearly bounded the previous paragraph guarantees that $g_1, \ldots, g_k$ are also bounded.

Finally we prove $(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$ in the general case. For each equivalence class E_i $(i = 1, \ldots, n)$ of $\{a_1, \ldots, a_k\}$ with respect to the relation $a \sim b \Leftrightarrow a/b \in \mathbb{Q}$ let h_i be the sum of those f_j-s for which $a_j \in E_i$. This way we get a decomposition $f = h_1 + \ldots + h_n$ such that each h_i is a b_i-periodic measurable $\mathbb{R} \to \mathbb{R}$ function and if every f_j is bounded then so is every h_i.

Since by (iv), $1/b_1, \ldots, 1/b_n$ are linearly independent over $\mathbb{Q}$, Theorem 2.3 implies that every $\{h_i\}$ is constant almost everywhere. By adding constants to some of the functions $f_1, \ldots, f_k$ we can guarantee that each $\{h_i\} = 0$ almost everywhere, which means that we can suppose that each h_i is almost everywhere integer valued. Since $h_i = \sum_{a_j \in E_i} f_j$ and $a_j/a_{j'} \in \mathbb{Q}$ if $a_j, a_{j'} \in E_i$, the first considered case can be applied for each h_i. □

3 Integer valued decompositions

It is natural to ask whether we can get (everywhere) integer valued decompositions in (i), (ii) and (iii) of Theorem 2.5 or not. We will see that this depends on the answers to some questions about the general (non-measurable) case.

Question 3.1 *Can one add the following statement to the list of equivalent statements of Theorem 2.5?*

(i'') *If an integer valued function $f : \mathbb{R} \to \mathbb{Z}$ can be decomposed as $f = f_1 + \ldots + f_k$ such that each f_j is an a_j-periodic measurable $\mathbb{R} \to \mathbb{R}$ function then f can be also decomposed as $f = g_1 + \ldots + g_k$ such that each g_j is an a_j-periodic integer valued measurable function.*

We shall prove that this question is actually equivalent with Question 0.2. For proving the equivalence of these questions we need the following lemma, which might be known.

Lemma 3.2 *For every $l = 1, 2, \ldots$ there exists an additive subgroup A_l of $\mathbb{R}$ such that*

(a) A_l is isomorphic to $\mathbb{Z}^l = \mathbb{Z} \times \ldots \times \mathbb{Z}$ and

(b) whenever $k \in \mathbb{N}$, $t_1, \ldots, t_k \in A_l \setminus \{0\}$ and $t_i/t_j \notin \mathbb{Q}$ for every $i \neq j$ then $\frac{1}{t_1}, \ldots, \frac{1}{t_k}$ are linearly independent over $\mathbb{Q}$.

Proof. We prove by induction. For $l = 1$ we can choose $A_1 = \mathbb{Z}$.

Now we construct A_{l+1} from A_l. For fixed $k \in \mathbb{N}$, $a = (a_1, \ldots, a_k) \in A_l^k$, $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ and $n = (n_1, \ldots, n_k) \in (\mathbb{Z} \setminus \{0\})^k$ let

$$\Phi_{a,m,n}(x) = \frac{n_1}{a_1 + m_1 x} + \ldots + \frac{n_k}{a_k + m_k x}.$$

For some polynomial $P_{a,m,n}(x)$, the function $\Phi_{a,m,n}(x)$ can be also written as

$$\Phi_{a,m,n}(x) = \frac{P_{a,m,n}(x)}{(a_1 + m_1 x) \cdot \ldots \cdot (a_k + m_k x)}.$$

Choose $y \in \mathbb{R}$ such that $y \notin A_l$ and y is not the root of any of those polynomials $P_{a,m,n}(x)$ that are not identically zero. This is possible since A_l is countable and we have only countably many polynomials and each has finitely many roots.

We claim that $A_{l+1} = A_l + \mathbb{Z}y$ has the required properties. Since $y \notin A_l$, A_{l+1} is indeed isomorphic to $\mathbb{Z}^{l+1}$. Suppose that $t_1, \ldots, t_k \in A_{l+1} \setminus \{0\}$ and $t_i/t_j \notin \mathbb{Q}$ for every $i \neq j$ but $\frac{1}{t_1}, \ldots, \frac{1}{t_k}$ are not linearly independent over $\mathbb{Q}$. Then there exists $a = (a_1, \ldots, a_k) \in A_l^k$ and $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ such that $t_i = a_i + m_i y$ for each i and there exists $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k \setminus \{0, \ldots, 0\}$ such that $\frac{n_1}{t_1} + \ldots + \frac{n_k}{t_k} = 0$. Then $\Phi_{a,m,n}(y) = 0$, so $P_{a,m,n}(y) = 0$. We can suppose that $n = (n_1, \ldots, n_k) \in (\mathbb{Z} \setminus \{0\})^k$ since we may simply omit every superfluous t_j. By the definition of y, this implies that $P_{a,m,n}$ is identically zero, so $\Phi_{a,m,n}(x) = 0$ for every x such that $a_i + m_i x \neq 0$ for every i.

We cannot have $m_1 = \ldots = m_k = 0$ since then we would have $t_1, \ldots, t_k \in A_l$, which would mean that A_l does not satisfy property (b). Let i be such that $m_i \neq 0$. Then on one hand

$$\lim_{x \to -\frac{a_i}{m_i}} \left| \frac{n_i}{a_i + m_i x} \right| = \infty,$$

on the other hand $\lim_{x \to -\frac{a_i}{m_i}} \Phi_{a,m,n} = 0$, so there must be an other term $\frac{n_j}{a_j + m_j x}$ of $\Phi_{a,m,n}$ with similar property. This means that there exists a $j \neq i$ such that $a_j/m_j = a_i/m_i$. But this implies that

$$\frac{t_i}{t_j} = \frac{a_i + m_i x}{a_j + m_j x} = \frac{m_i}{m_j} \in \mathbb{Q},$$

which is a contradiction. □

Proposition 3.3 *The answers to Question 3.1 and Question 0.2 must be the same.*

Proof. First suppose that the answer is affirmative to Question 3.1 and suppose that an integer valued function $f : \mathbb{R} \to \mathbb{Z}$ can be written as $f = g_1 + \ldots + g_k$, where each g_j is a real valued a_j-periodic function for some $a_j \in \mathbb{R}$. We want to show that f can be also written as $f = h_1 + \ldots + h_k$, where each h_j is an integer valued a_j-periodic function. It is enough to find such h_j-s on the additive group A generated by $a_1, \ldots, a_k$ since then we can define every h_j the same way on every coset of A. Then we can also suppose that $f, g_1, \ldots, g_k$ are defined also only on A, which is isomorphic to a group of the form $\mathbb{Z}^l$ for some l.

By Lemma 3.2 there exists an additive subgroup A_l of $\mathbb{R}$ that is isomorphic to $\mathbb{Z}^l$ (and so to A as well) and satisfies (b) of Lemma 3.2. Hence we may assume that $f, g_1, \ldots, g_k$ are defined on A_l. For every $x \in \mathbb{R}$ let

$$F(x) = \begin{cases} f(x) & \text{if } x \in A_l, \\ 0 & \text{if } x \notin A_l \end{cases}$$

and for each $j = 1, \ldots, k$,

$$G_j(x) = \begin{cases} g_j(x) & \text{if } x \in A_l, \\ 0 & \text{if } x \notin A_l. \end{cases}$$

Then for each j the function G_j is a_j-periodic and also measurable since A_j has measure 0. Since $a_1, \ldots, a_k \in A_l$ and A_l satisfies (b) of Lemma 3.2, (iv) of Theorem 2.5 holds for $a_1, \ldots, a_k$. On the other hand, affirmative answer to Question 3.1 means that (iv) of Theorem 2.5 implies (i'') of Question 3.1. Therefore we may apply (i'') of Question 3.1 for $F = G_1 + \ldots + G_k$ to get a decomposition $F = H_1 + \ldots + H_k$ such that each $H_j : \mathbb{R} \to \mathbb{Z}$ is a_j-periodic. Then the restriction h_j of H_j to A_l $(j = 1, \ldots, k)$ gives a suitable decomposition of f on A.

Now we suppose that the answer to Question 0.2 is affirmative and we prove that then (i'') of Question 3.1 is equivalent to $(i), (i'), (ii), (ii'), (iii), (iii')$ and (iv) of Theorem 2.5. Since in the $(i) \Rightarrow (iv)$ proof of Theorem 2.5 integer valued f is constructed the same argument also proves $(i'') \Rightarrow (iv)$. So it is enough to prove $(i) \Rightarrow (i'')$

Suppose that (i) holds, $f : \mathbb{R} \to \mathbb{Z}$, $f = f_1 + \ldots + f_k$ and each f_j is a measurable a_j-periodic function. By (i), there exists a decomposition $f = g_1 + \ldots + g_k$ such that each g_j is a measurable a_j-periodic almost everywhere integer valued function. All we have to do is replacing g_j-s by integer valued measurable functions.

Let

$$E_j = \{x \,:\, g_j(x) \notin \mathbb{Z}\} \quad \text{and} \quad E = \left(\cup_{j=1}^k E_j\right) + a_1\mathbb{Z} + \ldots + a_k\mathbb{Z}.$$

Then E is a set of measure zero and it is a_j-periodic for every j. Thus for each j the function $g_j\chi_{\mathbb{R}\setminus E}$ is a_j-periodic. Then, by the assumption that the answer to Question 0.2 is affirmative, $f\chi_E = g_1\chi_E + \ldots + g_k\chi_E$ implies that there exists a decomposition

$$f\chi_E = F_1 + \ldots + F_k$$

such that each F_j is an integer valued a_j-periodic function.

For each j let

$$G_j(x) = g_j \chi_{\mathbb{R}\setminus E} + F_j \chi_E.$$

Then G_j is clearly a_j-periodic and integer valued. It is also measurable since g_j is measurable and E is of measure zero. Since

$$f(x) = G_1(x) + \ldots + G_k(x)$$

clearly holds both for $x \in E$ and $x \in \mathbb{R} \setminus E$ we obtained a decomposition we wanted. $\square$

Now we prove that we cannot guarantee everywhere integer valued decompositions in *(ii)* and *(iii)* of Theorem 2.5.

Proposition 3.4 *There exists $a_1, a_2, a_3 \in \mathbb{R}$ such that $\frac{1}{a_1}, \frac{1}{a_2}$ and $\frac{1}{a_3}$ are linearly independent over $\mathbb{Q}$ and a function $f : \mathbb{R} \to \{0, 1\}$ that has a bounded measurable real valued (a_1, a_2, a_3)-periodic decomposition but does not have a bounded measurable integer valued (a_1, a_2, a_3)-periodic decomposition.*

Consequently one cannot replace "almost everywhere integer valued" by "integer valued" in (ii) and (iii) of Theorem 2.5.

Proof. Choose a_1, a_2 and a_3 so that $a_1 + a_2 = a_3$ but $\frac{1}{a_1}, \frac{1}{a_2}$ and $\frac{1}{a_3}$ are linearly independent over $\mathbb{Q}$, which is possible for example by taking $a_1, a_2 \in A_2$ such that $a_1/a_2 \notin \mathbb{Q}$, where A_2 is the additive subgroup of $\mathbb{R}$ obtained by Lemma 3.2.

By Theorem 0.3 there exists a function $u : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ that has a decomposition $u = u_1 + u_2 + u_3$ such that u_1, u_2 and u_3 are bounded real valued periodic functions with periods $(1, 0), (0, 1)$ and $(1, 1)$, respectively, but u has no decomposition $u = v_1 + v_2 + v_3$ such that v_1, v_2 and v_3 are bounded integer valued periodic functions with the same periods.

Let $E = a_1\mathbb{Z} + a_2\mathbb{Z}$,

$$f(x) = \begin{cases} u(n, m) & \text{if } x = na_1 + ma_2 \ (n, m \in \mathbb{Z}), \\ 0 & \text{if } x \notin E \end{cases}$$

and for each $j = 1, 2, 3$,

$$f_j(x) = \begin{cases} u_j(n, m) & \text{if } x = na_1 + ma_2 \ (n, m \in \mathbb{Z}), \\ 0 & \text{if } x \notin E. \end{cases}$$

Then clearly f maps to $\{0, 1\}$, $f = f_1 + f_2 + f_3$ and each f_j is a_j-periodic, bounded and measurable (since almost everywhere zero).

But f cannot have a decomposition $f = g_1 + g_2 + g_3$ such that each g_j is a_j-periodic, bounded and integer valued since then $v_j(n, m) = g_j(na_1 + ma_2)$ $(j = 1, 2, 3)$ would give an integer valued bounded decomposition of u with periods a_1, a_2, a_3, which is impossible. $\square$

Finally we pose two problems.

Problem 3.5 *Characterize those periods $a_1, \ldots, a_k \in \mathbb{R}\setminus\{0\}$ for which the existence of a bounded measurable real valued $(a_1, \ldots, a_k)$-periodic decomposition of an integer valued function implies the existence of a bounded measurable integer valued $(a_1, \ldots, a_k)$-periodic decomposition.*

Theorem 2.5 implies that (*iv*) of Theorem 2.5 is a necessary condition but Proposition 3.4 shows that it is not sufficient. The proofs of Propositions 3.1 and 3.4 indicate that this problem must be also related to the analogue non-measurable problem, which seems to be also open.

Problem 3.6 *Characterize those periods* $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$ *for which the existence of a bounded real valued* $(a_1, \dots, a_k)$*-periodic decomposition of an integer valued function implies the existence of a bounded integer valued* $(a_1, \dots, a_k)$*-periodic decomposition.*

As we mentioned after Theorem 0.3, it is proved in [8] that some restriction on the periods is necessary.

References

[1] B. Farkas, Sz. Gy. Révész, *Decomposition as the sum of invariant functions with respect to commuting transformations*, Aequationes Math, to appear.

[2] L. Fuchs, *Abelian groups*, Akadémiai Kiadó, Budapest, 1958.

[3] Z. Gajda, *Note on decomposition of bounded functions into the sum of periodic terms*, Acta Math. Hungar. **59** (1992), no. 1-2, 103–106.

[4] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1960.

[5] B. Farkas, T. Keleti, Sz. Gy. Révész, *Invariant decomposition of functions with respect to commuting invertible transformations*, submitted.

[6] V.M. Kadets, S.B. Shumyatskiy, *Averaging Technique in the Periodic Decomposition problem*, Mat. Fiz. Anal. Geom. **7** (2000), no. 2, 184–195.

[7] V.M. Kadets, S.B. Shumyatskiy, *Additions to the Periodic decomposition theorem*, Acta Math. Hungar. **90** (2001), no. 4, 293–305.

[8] Gy. Károlyi, G. Kós, T. Keleti, I. Z. Ruzsa *Periodic decomposition of integer valued functions*, Acta Math. Hung., to appear.

[9] T. Keleti *On the differences and sums of periodic measurable functions*, Acta Math. Hung. **75** (4) (1997), 279–286.

[10] T. Keleti, *Difference functions of periodic measurable functions*, Fund. Math. **157** (1998), 15–32.

[11] M. Laczkovich, Sz.Gy. Révész, *Periodic decompositions of continuous functions*, Acta Math. Hungar. **54** (1989), no. 3-4, 329–341.

[12] M. Laczkovich, Sz. Gy. Révész, *Decompositions into periodic functions belonging to a given Banach space*, Acta Math. Hung. **55** (3-4) (1990), 353–363.

[13] S. Mortola, R. Peirone, *The sum of periodic functions*, Boll. Un. Mat. Ital. **8** 2-B (1999), 393–396.

[14] T. Natkaniec, W. Wilczyński, *Sums of periodic Darboux functions and measurability*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 369–376.

[15] M. Wierdl, *Continuous functions that can be represented as the sum of finitely many periodic functions*, Mat. Lapok **32** (1984) 107–113 (in Hungarian).

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