Periodic Lip^α functions with Lip^β difference functions *

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December 21, 2004

1 Introduction

In [6] the following notion was introduced: Let $\mathbf{G}$ be either the additive subgroup of reals $\mathbf{R}$ or the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of functions on $\mathbf{G}$ with $\mathcal{F} \supset \mathcal{G}$. We denote by $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ the class of those subsets H of $\mathbf{G}$, for which a function $f \in \mathcal{F}$ can have difference functions $\Delta_h f(x) = f(x+h) - f(x)$ in $\mathcal{G}$ for every $h \in H$ without f belonging to $\mathcal{G}$. That is,

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \Big\{ H \subset \mathbf{G} : \exists f \in \mathcal{F} \setminus \mathcal{G} \;\; \Delta_h f \in \mathcal{G} \;\; \forall h \in H \Big\}.$$

We denote by Lip^α the class of functions f on $\mathbf{T}$ for which there exists an $L > 0$ such that $|f(x) - f(y)| \le L|x-y|^\alpha$ for every $x, y \in \mathbf{T}$. (Sometimes we identify $\mathbf{T}$ with $[0,1)$. If $a \in \mathbf{T}$ then by $|a|$ we mean $\min(a, 1-a)$.)

It was proved in [6] (Theorem 4.10) that for $0 < \alpha < \beta \le 1$ we have $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) \subset \mathfrak{F}_\sigma$ where $\mathfrak{F}_\sigma$ denotes the family of those subsets of $\mathbf{T}$ that can be covered by a proper F_σ subgroup of $\mathbf{T}$. Generalizing a result of M. Balcerzak, Z. Buczolich and M. Laczkovich [1], it was also proved in [6] (Theorem 5.3) that equality holds if $\beta = 1$. In this paper we investigate the case when $0 < \alpha < \beta < 1$.

A set $H \subset \mathbf{T}$ is called a *pseudo-Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $n_0(x)$ such that $|\sin q_n \pi x| < \varepsilon_n$ if $n \ge n_0(x)$. We denote the family of pseudo-Dirichlet sets by $p\mathfrak{D}$.

The pseudo-Dirichlet sets were considered by N. Bary [2] who proved that they are contained in the class of N_0-sets. Clearly $p\mathfrak{D}$ contains all the Dirichlet sets; in fact, pseudo-Dirichlet sets are the countable increasing unions of Dirichlet sets (see [**?**]). These notions together with several other notions of thinnes in harmonic analysis is discussed among others in [3] and [4].

In this paper we prove that

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta),$$

and that all the $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ classes are the same for any $0 < \alpha < \beta < 1$.

1991 *Mathematical Reviews Classification.* Primary 26A16; Secondary 26A33, 39A70, 42A16.

*The author wishes to express his gratitude to Professor Miklós Laczkovich for his advice and encouragement to the completion of a PhD dissertation [5] which this paper is based on.

†Supported by OTKA grant F 019468.

2 $p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$

We will use the following well known lemma (see e.g. [2] Chapter XI, pp. 691, Theorem 2):

Lemma 2.1 *If $0 < \gamma < 1$ and (q_n) is an increasing sequence of integers such that $q_{n+1}/q_n > \lambda$ for a suitable $\lambda > 1$, then for any sequence (a_n) of complex numbers,*

$$|a_n| = O(1/q_n^\gamma) \qquad \Longleftrightarrow \qquad g(x) = \sum_{n=1}^{\infty} a_n e^{2\pi i q_n x} \in \mathrm{Lip}^\gamma.$$

Theorem 2.2 *For any $0 < \alpha < \beta < 1$,*

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta).$$

That is, for any $0 < \alpha < \beta < 1$ and for any pseudo-Dirichlet set H, there exists a Lip^α function f such that $\Delta_h f$ is Lip^β for any $h \in H$ but f is not Lip^β.

Proof. Let H be a pseudo-Dirichlet set. Take a sequence $q_1 < q_2 < \ldots$ and a sequence $\varepsilon_n \to 0$ witnessing the pseudo-Dirichlet property of H. Selecting a suitable subsequence, we may assume that $q_{n+1} > 2q_n$ for every $n \in \mathbf{N}$. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^\beta \delta_n} e^{2\pi i q_n x}, \qquad \text{where} \qquad \delta_n = \max(\varepsilon_n, 1/q_n^{\beta-\alpha}).$$

Since $\frac{1}{q_n^\beta \delta_n} \leq 1/q_n^\alpha$ and $q_{n+1}/q_n > 2$ $(n \in \mathbf{N})$, we can apply Lemma 2.1 to obtain $f \in \mathrm{Lip}^\alpha$. On the other hand, since $\delta_n \to 0$, $\frac{1}{q_n^\beta \delta_n} \neq O(1/q_n^\beta)$, so Lemma 2.1 implies that $f \notin \mathrm{Lip}^\beta$.

For an $h \in H$,

$$\Delta_h f(x) = \sum_{n=1}^{\infty} \frac{1}{q_n^\beta \delta_n} \left(e^{2\pi i q_n h} - 1\right) e^{2\pi i q_n x}$$

and

$$\left| \frac{1}{q_n^\beta \delta_n} \left(e^{2\pi i q_n h} - 1\right) \right| = \frac{1}{q_n^\beta \delta_n} 2|\sin \pi q_n h| \leq \frac{1}{q_n^\beta \delta_n} 2\varepsilon_n \leq \frac{2}{q_n^\beta},$$

therefore, by Lemma 2.1, $\Delta_h f \in \mathrm{Lip}^\beta$. □

Combining the previous theorem with the result of [6] mentioned in the Introduction, we get the following:

Corollary 2.3 *For any $0 < \alpha < \beta < 1$,*

$$p\mathfrak{D} \subset \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) \subset \mathfrak{F}_\sigma.$$

3 The classes $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ are the same for $0 < \alpha < \beta < 1$

The following easy lemmas were obtained in [6]:

Lemma 3.1 (Monotonicity Lemma) *If* $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{G}$ *then* $\mathfrak{H}(\mathcal{F}_1, \mathcal{G}) \supset \mathfrak{H}(\mathcal{F}_2, \mathcal{G})$.

Lemma 3.2 *If* $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ *and* $\mathfrak{H}(\mathcal{F}_1, \mathcal{F}_2) \subset \mathfrak{H}(\mathcal{F}_2, \mathcal{F}_3)$ *then* $\mathfrak{H}(\mathcal{F}_1, \mathcal{F}_3) = \mathfrak{H}(\mathcal{F}_2, \mathcal{F}_3)$.

We will need the notion of fractional integration. There are several different notions of fractional integrals (see e.g. the monograph [7]); here we use the so called Weyl fractional integral which is defined in the following way (see [7] p. 263 or [8] Vol. II p. 133):

Let f be an integrable function on $\mathbf{T}$ and suppose that $\int_{\mathbf{T}} f = 0$. Then, for any $\gamma > 0$ let

(1) $$I_\gamma[f](x) = \int_{\mathbf{T}} f(t)\Psi_\gamma(x-t)dt,$$

where

(2) $$\Psi_\gamma(t) = \sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{e^{2\pi int}}{(2\pi in)^\gamma}.$$

It is known (see e.g. [8]) that the series in (2) converges everywhere on $\mathbf{T}\setminus\{0\}$ and the integral in (1) exists almost everywhere. (If $f \sim \sum c_n e^{2\pi inx}$ then $I_\gamma[f] \sim \sum c_n \frac{e^{2\pi inx}}{(2\pi in)^\gamma}$.)

Since the operator I_γ is defined by a convolution it commutes with the translation operator and it is linear; that is,

(3) $$I_\gamma[f(y+h)](x) = I_\gamma[f(y)](x+h),$$

and

(4) $$I_\gamma[cf + dg] = cI_\gamma[f] + dI_\gamma[g].$$

Denote by Lip_0^λ the class of the functions of Lip^λ with integral 0 (over $\mathbf{T}$). It is also well known (see e.g. [7] p. 275) that if $\gamma, \lambda > 0$ and $\gamma + \lambda < 1$, then I_γ is a bijection (actually it is an isomorphism) between the classes Lip_0^λ and $\mathrm{Lip}_0^{\lambda+\gamma}$; that is,

(5) $$I_\gamma : \ \mathrm{Lip}_0^\lambda \leftrightarrow \mathrm{Lip}_0^{\lambda+\gamma} \qquad (\lambda + \gamma < 1).$$

Theorem 3.3 *For any* $0 < \alpha_1 < \beta_1 < 1$ and $0 < \alpha_2 < \beta_2 < 1$,

$$\mathfrak{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathfrak{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2}).$$

Proof. First we prove that if $0 < \alpha < \beta$ and $\beta + \gamma < 1$ then

$$\mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma}) = \mathfrak{H}(\mathrm{Lip}^{\alpha}, \mathrm{Lip}^{\beta}). \tag{6}$$

Indeed, (3) and (4) implies that the operator I_γ also commutes with the difference operator Δ_h; that is

$$\Delta_h I_\gamma[f] = I_\gamma[\Delta_h f]. \tag{7}$$

It follows from (5) and (7) that if $f_0 \in \mathrm{Lip}_0^\alpha \setminus \mathrm{Lip}_0^\beta$ and $\Delta_h f_0 \in \mathrm{Lip}_0^\beta$ for every $h \in H$ then $I_\gamma[f_0] \in \mathrm{Lip}_0^{\alpha+\gamma} \setminus \mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h I_\gamma[f_0] \in \mathrm{Lip}_0^{\beta+\gamma}$ for every $h \in H$. Furthermore, if $g_0 \in \mathrm{Lip}_0^{\alpha+\gamma} \setminus \mathrm{Lip}_0^{\beta+\gamma}$ and $\Delta_h g_0 \in \mathrm{Lip}_0^{\beta+\gamma}$ for every $h \in H$ then $I_\gamma^{-1}[g_0] \in \mathrm{Lip}_0^\alpha \setminus \mathrm{Lip}_0^\beta$ and $\Delta_h I_\gamma^{-1}[g_0] \in \mathrm{Lip}_0^\beta$ for every $h \in H$. Therefore if the function $f : \mathbf{T} \to \mathbf{R}$ witnesses that $H \in \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ then $I_\gamma[f_0]$ - where $f_0 = f - \int_{\mathbf{T}} f$ - witnesses that $H \in \mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma})$; furthermore if the function $g : \mathbf{T} \to \mathbf{R}$ witnesses that $H \in \mathfrak{H}(\mathrm{Lip}^{\alpha+\gamma}, \mathrm{Lip}^{\beta+\gamma})$ then $I_\gamma^{-1}[g_0]$ - where $g_0(x) = g - \int_{\mathbf{T}} g$ - witnesses that $H \in \mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$.

Now we prove that for any $0 < \eta < \delta < \beta < 1$,

$$\mathfrak{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta}) = \mathfrak{H}(\mathrm{Lip}^{\beta-\eta}, \mathrm{Lip}^{\beta}). \tag{8}$$

Indeed, by (7), $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta-\delta/2}) = \mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2}, \mathrm{Lip}^{\beta})$, which implies, by Lemma 3.2, that $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta}) = \mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2}, \mathrm{Lip}^{\beta})$. Thus we have also $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta}) = \mathfrak{H}(\mathrm{Lip}^{\beta-\delta/2^k}, \mathrm{Lip}^{\beta})$ for any $k \in \mathbf{N}$. Then, by the Monotonicity Lemma, $\mathfrak{H}(\mathrm{Lip}^{\beta-\delta}, \mathrm{Lip}^{\beta}) = \mathfrak{H}(\mathrm{Lip}^{\beta-\eta}, \mathrm{Lip}^{\beta})$.

Finally, supposing that $\beta_1 \le \beta_2$ and applying (7) and (8), we get

$$\mathfrak{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1}) = \mathfrak{H}(\mathrm{Lip}^{\alpha_1+\beta_2-\beta_1}, \mathrm{Lip}^{\beta_2}) = \mathfrak{H}(\mathrm{Lip}^{\alpha_2}, \mathrm{Lip}^{\beta_2}). \quad \square$$

Remark 3.4 Unfortunately this proof does not work if β_1 or β_2 equals 1. Namely, (5) is not true for $\lambda + \gamma = 1$. In this case I_γ is a bijection between $\mathrm{Lip}_0^{1-\gamma}$ and Λ_{*0}, the class of Zygmund functions on $\mathbf{T}$ with 0 integral.

(A function f is Zygmund, if for any x and h, $|f(x+h) - 2f(x) + f(x-h)| \le Ch$. The class of Zygmund functions is denoted by Λ_*. It is known (see e.g. [8] Vol. I p. 43-44 and Vol. II p. 138) that

$$\mathrm{Lip}^1(\mathbf{R}) \subset \Lambda_* \subset \mathrm{Lip}^\alpha(\mathbf{R}) \quad \forall 0 < \alpha < 1,$$

and $\Lambda_* \ne \mathrm{Lip}^1(\mathbf{R})$.)

Therefore with this method we can only prove that

$$\mathfrak{H}(\mathrm{Lip}^\alpha, \Lambda_*(\mathbf{T})) = \mathfrak{H}(\mathrm{Lip}^{\alpha_1}, \mathrm{Lip}^{\beta_1})$$

for any $0 < \alpha < 1$ and $0 < \alpha_1 < \beta_1 < 1$.

However, if, for a fixed $0 < \alpha < 1$, one can find a linear operator I that commutes with the translation operator and I is a bijection between Lip^1 and Lip^α (or between Lip^1 and Λ_*) then Theorem 3.3 would remain true for $\beta_1 = 1$. This would give a complete answer to our question since, as we mentioned in the Introduction, it was proved in [6] that $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^1) = \mathfrak{F}_\sigma$, so the existence of such an operator would imply that $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) = \mathfrak{F}_\sigma$ for any $0 < \alpha < \beta \le 1$.

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