

Periodic decomposition of integer valued functions

Gyula Károlyi*, Tamás Keleti†, Géza Kós‡ and Imre Z. Ruzsa§

Abstract

We study those functions that can be written as a finite sum of periodic integer valued functions. On \mathbb{Z} we give three different characterizations of these functions. For this we prove that the existence of a real valued periodic decomposition of a $\mathbb{Z} \rightarrow \mathbb{Z}$ function implies the existence of an integer valued periodic decomposition with the same periods. This result depends on the representation of the greatest common divisor of certain polynomials with integer coefficients as a linear combination of the given polynomials where the coefficients also belong to $\mathbb{Z}[x]$. We give an example of an $\mathbb{R} \rightarrow \{0, 1\}$ function that has a bounded real valued periodic decomposition but does not have a bounded integer valued periodic decomposition with the same periods. It follows that the class of bounded $\mathbb{Z} \rightarrow \mathbb{Z}$ functions has the decomposition property as opposed to the class of bounded $\mathbb{R} \rightarrow \mathbb{Z}$ functions. If the periods are pairwise commensurable or not prescribed, then we get more general results.

Introduction

Denote by Δ_a the *difference operator* $\Delta_a f(x) = f(x+a) - f(x)$. A function f is a -periodic, if $\Delta_a f = 0$. We say that a function f has an (a_1, \dots, a_k) -*decomposition*, if it can be represented as $f = f_1 + \dots + f_k$, where f_j is an a_j -periodic function for each $1 \leq j \leq k$. If each f_j has a certain property, then we say that the decomposition has this property.

The characterization of functions having (a_1, \dots, a_k) -decompositions with some given property for a fixed k -tuple (a_1, \dots, a_k) has been studied in many

*Supported by Hungarian Scientific Research Foundation (OTKA) grants no. T 43631 and NK 67867.

†Supported by OTKA grants F 43620 and T 49786. This research was initiated when the author was on leave at the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences.

‡Supported by János Bolyai Research Fellowship.

§Supported by OTKA grants T 43623 and K 61908.

MSC code: 39A10; Secondary 11B99, 13F20, 26A99, 39B52

Key Words: periodic functions, periodic decomposition, integer valued and real valued functions, difference operator, decomposition property, commutative groups, torsion-free, greatest common divisor of polynomials with integer coefficients, resultant, cyclotomic polynomials

papers, among others in [2], [4], [5], [6], [7], [8], [11], [12], [13], [14] and [15]. Since the difference operators are linear and they commute, the condition

$$\Delta_{a_1} \dots \Delta_{a_k} f = 0 \tag{1}$$

is clearly necessary for having any (a_1, \dots, a_k) -decomposition.

A class of functions \mathcal{F} is said to have the *decomposition property*, if every $f \in \mathcal{F}$ that satisfies (1) has an (a_1, \dots, a_k) -decomposition in \mathcal{F} . Since the identity function $f(x) = x$ satisfies $\Delta_1 \Delta_1 f = 0$, but it is not the sum of two 1-periodic functions, many natural classes of functions (e.g. all $\mathbb{R} \rightarrow \mathbb{R}$ functions, continuous $\mathbb{R} \rightarrow \mathbb{R}$ functions) fail to have the decomposition property. However, many classes of functions do have the decomposition property: for example the class of all bounded continuous $\mathbb{R} \rightarrow \mathbb{R}$ functions [11], the class of all bounded $\mathbb{R} \rightarrow \mathbb{R}$ functions [4], [12], the class of all bounded measurable $\mathbb{R} \rightarrow \mathbb{R}$ functions [12], and the class of all bounded real valued functions on an arbitrary commutative group [12].

In all of the above mentioned papers real valued functions were studied. In this paper the focus is on integer valued functions, and we do not assume measurability. We prove among others that the class of bounded $\mathbb{Z} \rightarrow \mathbb{Z}$ functions has the decomposition property (Corollary 2.2), but the class of bounded $\mathbb{R} \rightarrow \mathbb{Z}$ functions does not have the decomposition property (Corollary 3.4). In fact, we can characterize those torsion-free commutative groups on which the class of all bounded integer valued functions has the decomposition property (Corollary 3.5).

For the above results we study the natural question, whether the existence of a real valued (bounded) periodic decomposition of an integer valued function implies the existence of an integer valued (bounded) periodic decomposition. The answer is positive for functions on \mathbb{Z} (Theorem 2.1) and negative for bounded functions on \mathbb{R} (Corollary 3.3).

If we allow different periods, then we get a positive answer (Corollary 4.7) on any torsion-free commutative group. Moreover, we obtain an integer valued decomposition even with the original periods, in case these are either commensurable (so in particular on \mathbb{Z}) or pairwise incommensurable (Theorem 4.9). Several natural questions remain open. Measurable integer valued periodic decompositions of functions are studied in a subsequent paper [9] of the second author.

The result on the decomposition of $\mathbb{Z} \rightarrow \mathbb{Z}$ functions (Theorem 2.1) depends on the representation of the greatest common divisor of certain polynomials with integer coefficients as a linear combination of the given polynomials where the coefficients also belong to $\mathbb{Z}[x]$. Due to the general lack of a division algorithm in $\mathbb{Z}[x]$, the existence of the required decomposition is far from being obvious, and actually fails in general. We elaborate on this in the next section, giving a necessary and sufficient condition in the case of polynomials that can be written as the product of cyclotomic polynomials.

1 On the number theory of polynomials

Recall that, for any field F , the polynomial ring $F[x]$ is a unique factorization domain. In fact, it is a principle ideal domain; as a consequence of the euclidean

algorithm any number of polynomials have a greatest common divisor that is unique up to a unit factor, and can be expressed as a combination of the given polynomials. Although the euclidean algorithm fails in $\mathbb{Z}[x]$, it still has unique factorization, and thus any number of polynomials in $\mathbb{Z}[x]$ have a greatest common divisor that is unique up to a ± 1 factor. If $f_1, \dots, f_n \in F[x]$ are not all zero, then among their greatest common divisors there is a unique monic polynomial that can be referred to as ‘the’ greatest common divisor of the polynomials f_i , and is denoted by (f_1, \dots, f_n) . Similarly, if $f_1, \dots, f_n \in \mathbb{Z}[x]$, then (f_1, \dots, f_n) can be distinguished by the requirement of a positive leading coefficient. Note that if $\mathbb{Z} \subset F$, then (f_1, \dots, f_n) in $\mathbb{Z}[x]$ may differ from that in $F[x]$ (by a constant factor), but if all polynomials are monic, then it clearly cannot happen. In general, if $f, g \in \mathbb{Z}[x]$ are monic polynomials, then f divides g in $\mathbb{Z}[x]$ if and only if the same relation holds in $F[x]$, thus in such a situation the meaning of $f \mid g$ must be clear without any ambiguity. All this is standard and well-known, as well as the rest of the algebraic terminology we use in the sequel and can be found for example in the classical textbook of van der Waerden [16].

For $f, g \in F[x]$, we denote by $R(f, g)$ the resultant of the polynomials f and g . Obtained as the determinant of the so-called Sylvester matrix all whose entries are coefficients of f and g , it is an element of F that belongs to the subring generated by the coefficients of f and g . For monic polynomials f and g it can be expressed by the product formula

$$R(f, g) = \prod_{i=1}^d \prod_{j=1}^e (\alpha_i - \beta_j), \quad (2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_d$ and $\beta_1, \beta_2, \dots, \beta_e$ denote the roots (with multiplicity) of the polynomials f and g , respectively, in the algebraic closure of F . Thus, $(f, g) = 1$ if and only if $R(f, g) \neq 0$, both conditions being independent of the actual choice of the field F that contains the coefficients of f and g . The following statement may be folklore, but we were not able to find any references.

Theorem 1.1 *Suppose that f_1, f_2 are monic polynomials in $\mathbb{Z}[x]$ without a common factor. Then there exist polynomials $g_1, g_2 \in \mathbb{Z}[x]$ such that $f_1 g_1 + f_2 g_2 = 1$ if and only if $R(f_1, f_2) = \pm 1$.*

Proof. The condition $R(f_1, f_2) = \pm 1$ is sufficient since for arbitrary polynomials $f_1, f_2 \in \mathbb{Z}[x]$ there exist polynomials $g_1, g_2 \in \mathbb{Z}[x]$ such that $f_1 g_1 + f_2 g_2 = R(f_1, f_2)$, cf. [16], pp. 105–106. To see the necessity, assume that $R(f_1, f_2) \neq 0$ is divisible by a prime number p , and for some polynomials $g_1, g_2 \in \mathbb{Z}[x]$ and a constant $c \in \mathbb{Z}$ we have $f_1 g_1 + f_2 g_2 = c$. We prove that p must divide c , hence $c = 1$ is not possible. If we reduce the coefficients modulo p , then we find that f_1 and f_2 , viewed as polynomials over the Galois field $GF(p)$, have a common factor ψ in $GF(p)[x]$. Understanding ψ as a polynomial in $\mathbb{Z}[x]$ we can thus write in $\mathbb{Z}[x]$

$$c = f_1 g_1 + f_2 g_2 = \psi h_1 + h_2,$$

where h_1, h_2 are polynomials in $\mathbb{Z}[x]$ such that all coefficients of h_2 are divisible by p . To prove our statement it is enough to check that the coefficients of h_1 are also divisible by p . Assume that this is not case. Note that the degree of $\psi \in \mathbb{Z}[x]$ is not zero, and its leading coefficient is not divisible by p . If the degree of ψ is d , and in $h_1(x) = a_n x^n + \dots + a_0$ the largest index i with $p \nmid a_i$ is

denoted by t , then in ψh_1 the coefficient of x^{d+t} is not divisible by p , whereas in $c - h_2$, due to $d+t > 0$, the same coefficient is divisible by p , a contradiction. \square

In the present section we study polynomials that are products of cyclotomic polynomials, such functions being instrumental in the proof of Theorem 2.1. The m th cyclotomic polynomial Φ_m is the monic polynomial of degree $\varphi(m)$ whose roots are the m th primitive roots of unity $e^{2k\pi i/m}$, $1 \leq k \leq m$, $(k, m) = 1$. It is well-known that Φ_m has integer coefficients and is irreducible over \mathbb{Q} . Moreover,

$$x^m - 1 = \prod_{d|m} \Phi_d(x). \quad (3)$$

The main result of this section is the following.

Theorem 1.2 *Let F_i be finite subsets of \mathbb{N} such that $1 \notin F_i$ for $1 \leq i \leq k$. Suppose that $m \in F_i$, $m' \in F_j$, $m \mid m'$ implies $m' \in F_i$ for every pair i, j with $1 \leq i \neq j \leq k$. Then the greatest common divisor (f_1, f_2, \dots, f_k) of the polynomials $f_i = \prod_{m \in F_i} \Phi_m$ can be represented as $q_1 f_1 + q_2 f_2 + \dots + q_k f_k$ with suitable polynomials $q_i \in \mathbb{Z}[x]$.*

Proof. The statement is obvious if $k = 1$. Assume that $k = 2$, then

$$f = (f_1, f_2) = \prod_{m \in F_1 \cap F_2} \Phi_m \in \mathbb{Z}[x].$$

Consider $f'_i = f_i/f = \prod_{m \in F'_i} \Phi_m$, where $F'_i = F_i \setminus (F_1 \cap F_2)$, then $(f'_1, f'_2) = 1$. Here $F'_1 \cap F'_2 = \emptyset$, moreover if $i \neq j$, $m \in F'_i$ and $m \mid m'$, then $m' \notin F'_j$. If there exist polynomials $q_1, q_2 \in \mathbb{Z}[x]$ such that $1 = q_1 f'_1 + q_2 f'_2$, then $f = q_1 f_1 + q_2 f_2$, thus the theorem in the case $k = 2$ can be reduced to the following lemma.

Lemma 1.3 *Consider the polynomials $f_1 = \prod_{i=1}^s \Phi_{m_i}$ and $f_2 = \prod_{j=1}^t \Phi_{n_j}$, where m_i, n_j are positive integers. There exist polynomials $q_1, q_2 \in \mathbb{Z}[x]$ such that $q_1 f_1 + q_2 f_2 = 1$ if and only if for arbitrary integers i, j, c and prime number p we have $n_j/m_i \neq p^c$.*

Proof. f_1, f_2 are monic polynomials that have a common factor if and only if $n_j/m_i = 1$ for some $1 \leq i \leq s$, $1 \leq j \leq t$; in that case $q_1 f_1 + q_2 f_2 = 1$ cannot happen. Assume therefore that $(f_1, f_2) = 1$. The product formula (2) implies

$$R(f_1, g_1) = \prod_{i=1}^s \prod_{j=1}^t R(\Phi_{m_i}, \Phi_{n_j}).$$

According to a result of Apostol [1], for positive integers $m \neq n$ we have $R(\Phi_m, \Phi_n) = 1$, unless there is a prime p such that either m/n or n/m is a power of p , in which case the resultant is a power of p . The lemma then follows from Theorem 1.1. \square

To complete the proof of Theorem 1.2, assume that $k \geq 3$ and the result has been already verified for smaller values of k . There exist polynomials $q'_1, q'_2 \in \mathbb{Z}[x]$ such that $f = (f_{k-1}, f_k) = q'_1 f_{k-1} + q'_2 f_k$. Write $F = F_{k-1} \cap F_k$. It is easy

to check for any $1 \leq i \leq k-2$, that $m \in F_i$, $m' \in F$, $m \mid m'$ implies $m' \in F_i$, and $m \in F$, $m' \in F_i$, $m \mid m'$ implies $m' \in F$. Since $f = \prod_{m \in F} \Phi_m$, there exist polynomials $q_1, q_2, \dots, q_{k-2}, q \in \mathbb{Z}[x]$ such that

$$(f_1, f_2, \dots, f_k) = (f_1, f_2, \dots, f_{k-2}, f) = q_1 f_1 + q_2 f_2 + \dots + q_{k-2} f_{k-2} + q f.$$

Choosing $q_{k-1} = q q_1'$ and $q_k = q q_2'$ we obtain the desired representation of (f_1, f_2, \dots, f_k) . \square

Corollary 1.4 *Suppose that the positive integers a_1, a_2, \dots, a_k divide the positive integer n , and let $f_i(x) = 1 + x^{a_i} + x^{2a_i} + \dots + x^{n-a_i}$ for $1 \leq i \leq k$. Then there exist polynomials $q_i \in \mathbb{Z}[x]$ such that the greatest common divisor of the polynomials f_i can be expressed as*

$$(f_1, f_2, \dots, f_k) = q_1 f_1 + q_2 f_2 + \dots + q_k f_k.$$

Proof. According to (3) we can write the polynomials f_i as

$$f_i(x) = \frac{x^n - 1}{x^{a_i} - 1} = \prod_{m \in F_i} \Phi_m(x),$$

and it is readily checked that the sets $F_i = \{m \in \mathbb{N} : m \mid n, m \nmid a_i\}$ satisfy the conditions of Theorem 1.2. \square

A more direct proof of this corollary can be obtained as follows. Put $p_i(x) = x^{a_i} - 1$ and $q(x) = x^n - 1$. Let $r = [p_1, p_2, \dots, p_k]$ denote the least common multiple of the polynomials p_1, p_2, \dots, p_k , whereas $p = (p_1, p_2, \dots, p_k)$. Since as a simple consequence of unique factorization we have

$$(f_1, f_2, \dots, f_k) = \left(\frac{q}{p_1}, \frac{q}{p_2}, \dots, \frac{q}{p_k} \right) = \frac{q}{r},$$

it is enough to prove that the rational function $1/r$ belongs to the $\mathbb{Z}[x]$ -module

$$\mathcal{M} = \frac{1}{p_1} \mathbb{Z}[x] + \frac{1}{p_2} \mathbb{Z}[x] + \dots + \frac{1}{p_k} \mathbb{Z}[x].$$

This we prove by induction on k . The initial step being obvious, assume that $k \geq 2$, and the statement has been proven for $k-1$. Accordingly,

$$\frac{1}{r_{k-1}} := \frac{1}{[p_1, \dots, p_{k-1}]} \in \mathcal{M}.$$

Thus,

$$\frac{p_k}{rp} = \frac{r_{k-1} p_k}{rp} \cdot \frac{1}{r_{k-1}} = \frac{r_{k-1}, p_k}{rp} \cdot \frac{1}{r_{k-1}} = \frac{(r_{k-1}, p_k)}{p} \cdot \frac{1}{r_{k-1}} \in \mathcal{M}.$$

By symmetry, we have $p_i/rp \in \mathcal{M}$ for every $1 \leq i \leq k$.

Lemma 1.5 *There exist polynomials $c_1, c_2, \dots, c_k \in \mathbb{Z}[x]$ such that*

$$p = c_1 p_1 + c_2 p_2 + \dots + c_k p_k.$$

Proof. The factorization formula (3) immediately implies that $p(x) = x^d - 1$, where $d = (a_1, a_2, \dots, a_k)$. Alternatively, the same conclusion can be obtained by a repeated application of the euclidean algorithm, which in this special case can be fully carried out but with somewhat more tedious details. It is a routine exercise to show that there exist positive integers l_1, l_2, \dots, l_k such that $d = l_1 a_1 - l_2 a_2 - \dots - l_k a_k$. Since

$$x^d - 1 = (x^{l_1 a_1} - 1) - \sum_{i=2}^k x^{l_{i+1} a_{i+1} + \dots + l_k a_k + d} (x^{l_i a_i} - 1),$$

the statement follows with

$$c_1(x) = \frac{x^{l_1 a_1} - 1}{x^{a_1} - 1}, \quad c_i(x) = -(x^{l_{i+1} a_{i+1} + \dots + l_k a_k + d}) \frac{x^{l_i a_i} - 1}{x^{a_i} - 1} \quad (2 \leq i \leq k).$$

□

In view of this lemma,

$$\frac{1}{r} = \frac{p}{rp} = \sum_{i=1}^k c_i \frac{p_i}{rp} \in \mathcal{M},$$

and the proof is complete.

2 A positive result about functions on \mathbb{Z}

First we show that if an integer valued function on \mathbb{Z} has a real valued (a_1, \dots, a_k) -decomposition, then it also has an integer valued (a_1, \dots, a_k) -decomposition.

Theorem 2.1 *Suppose that an integer valued function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ can be written as $f = g_1 + \dots + g_k$, where each g_j is a real valued a_j -periodic function for some $a_j \in \mathbb{Z}$. Then f can be also written as $f = h_1 + \dots + h_k$, where each h_j is an integer valued a_j -periodic function.*

Proof. We can clearly suppose that $a_1, \dots, a_k > 0$. Let n denote the least common multiple of a_1, \dots, a_k .

Consider the polynomial $g(z) = \sum_{l=0}^{n-1} f(l)z^l \in \mathbb{Z}[z]$. Using that $f = \sum_{j=1}^k g_j$ and each g_j is a_j -periodic we can rewrite g in the following way.

$$\begin{aligned} g(z) &= \sum_{l=0}^{n-1} f(l)z^l \\ &= \sum_{j=1}^k \sum_{l=0}^{n-1} g_j(l)z^l \\ &= \sum_{j=1}^k \sum_{r=0}^{a_j-1} \sum_{u=0}^{n/a_j-1} g_j(ua_j + r)z^{ua_j+r} \\ &= \sum_{j=1}^k \sum_{r=0}^{a_j-1} g_j(r)z^r (1 + z^{a_j} + z^{2a_j} + \dots + z^{n-a_j}). \end{aligned} \quad (4)$$

Let $\Psi \in \mathbb{Z}[z]$ be the greatest common divisor of the polynomials

$$1 + z^{a_j} + z^{2a_j} + \dots + z^{n-a_j} \quad (j = 1, \dots, k).$$

By Corollary 1.4 we may express Ψ as

$$\Psi(z) = \sum_{j=1}^k q_j(z) \cdot (1 + z^{a_j} + z^{2a_j} + \dots + z^{n-a_j})$$

with suitable polynomials $q_1, \dots, q_k \in \mathbb{Z}[z]$.

By (4), Ψ is a divisor of g in $\mathbb{R}[z]$. Since $\Psi, g \in \mathbb{Z}[z]$ and Ψ is monic, the division algorithm implies that $p = g/\Psi \in \mathbb{Z}[z]$. Therefore

$$\sum_{l=0}^{n-1} f(l)z^l = g(z) = p(z)\Psi(z) = \sum_{j=1}^k p(z)q_j(z) \cdot (1 + z^{a_j} + z^{2a_j} + \dots + z^{n-a_j}).$$

For each $j = 1, \dots, k$ and $l = 0, 1, \dots, n-1$, let $h_j(l)$ be the sum of the coefficients of $z^l, z^{l+n}, z^{l+2n}, \dots$ in the polynomial

$$p(z)q_j(z) \cdot (1 + z^{a_j} + z^{2a_j} + \dots + z^{n-a_j}).$$

Extending each h_j to an n -periodic function on \mathbb{Z} one finds that each h_j is an integer valued a_j -periodic function such that $f = h_1 + \dots + h_k$. \square

Corollary 2.2 *The class of bounded $\mathbb{Z} \rightarrow \mathbb{Z}$ functions has the decomposition property.*

Proof. Let f be a bounded $\mathbb{Z} \rightarrow \mathbb{Z}$ function such that $\Delta_{a_1} \dots \Delta_{a_k} f = 0$. By a result of Laczkovich and Révész [12], for any commutative group A , the class of bounded $A \rightarrow \mathbb{R}$ functions has the decomposition property. Thus we can express f as $f = g_1 + \dots + g_k$ such that each g_j is an a_j -periodic $\mathbb{Z} \rightarrow \mathbb{R}$ function. In view of Theorem 2.1, we can also write f as $f = h_1 + \dots + h_k$ such that each h_j is an a_j -periodic $\mathbb{Z} \rightarrow \mathbb{Z}$ function. Since periodic functions on \mathbb{Z} are always bounded, this completes the proof. \square

Corollary 2.3 *Let A be an additive subgroup of \mathbb{Q} and $a_1, \dots, a_k \in A$. If a function $f : A \rightarrow \mathbb{Z}$ has a real valued (a_1, \dots, a_k) -decomposition, then it also has an integer valued (a_1, \dots, a_k) -decomposition.*

Proof. Let d be a common multiple of the denominators of a_1, \dots, a_k , and let $D = \{n/d : n \in \mathbb{Z}\}$. Since D is isomorphic to \mathbb{Z} , we can apply Theorem 2.1 on each coset $D + q$ ($q \in A$). \square

Corollary 2.4 *For any additive subgroup A of \mathbb{Q} , the class of bounded $A \rightarrow \mathbb{Z}$ functions has the decomposition property.* \square

Theorem 2.1 and Corollary 2.2 give two different characterizations of those $\mathbb{Z} \rightarrow \mathbb{Z}$ functions that have an integer valued (a_1, \dots, a_k) -decomposition for fixed $a_1, \dots, a_k \in \mathbb{Z} \setminus \{0\}$. Let n denote a fixed common multiple of a_1, \dots, a_k . To obtain a third characterization, we use the fact (see e.g. [10, Fact 2.8.]) that

a function $f : \mathbb{Z}_n \rightarrow \mathbb{C}$ has a complex valued (a_1, \dots, a_k) -decomposition if and only if

$$\{l \in \mathbb{Z} : \widehat{f}(l) \neq 0\} \subset \frac{n}{a_1}\mathbb{Z} \cup \dots \cup \frac{n}{a_k}\mathbb{Z}.$$

Here \widehat{f} denotes the discrete Fourier transform of f on \mathbb{Z}_n ; that is,

$$\widehat{f}(l) = \sum_{j=0}^{n-1} f(j)e^{(2\pi i/n)lj} \quad (l \in \mathbb{Z}).$$

Note that if f is real valued then, by taking the real part of each term, we can get a real valued decomposition from a complex valued decomposition. If f is integer valued, then $\widehat{f}(l) = 0$ if and only if $\widehat{f}((n, l)) = 0$, since $e^{2l\pi i/n}$ is a root of a polynomial $p \in \mathbb{Z}[x]$ if and only if $\Phi_{n/(n, l)}$ divides p , and the same holds for $e^{2(n, l)\pi i/n}$. Accordingly, a function $\mathbb{Z}_n \rightarrow \mathbb{Z}$ has a real valued (a_1, \dots, a_k) -decomposition if and only if

$$\{l \mid n : \widehat{f}(l) \neq 0\} \subset \frac{n}{a_1}\mathbb{Z} \cup \dots \cup \frac{n}{a_k}\mathbb{Z}. \quad (5)$$

Summarizing the various characterizations, we get the following.

Corollary 2.5 *For arbitrary numbers $a_1, \dots, a_k \in \mathbb{Z} \setminus \{0\}$ with a fixed common multiple n and a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$, the following four statements are equivalent:*

- (i) f has an integer valued (a_1, \dots, a_k) -decomposition;
- (ii) f has a real valued (a_1, \dots, a_k) -decomposition;
- (iii) f is bounded and $\Delta_{a_1} \dots \Delta_{a_k} f = 0$;
- (iv) f is an n -periodic function satisfying (5). □

Corollary 2.6 *An n -periodic function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ can be written as the sum of periodic integer valued functions with smaller periods if and only if $\widehat{f}(1) = 0$; that is, if*

$$\sum_{j=0}^{n-1} f(j)e^{(2\pi i/n)j} = 0.$$

Proof. If $\widehat{f}(1) = 0$, then let p_1, \dots, p_k be the prime divisors of n and let $a_j = \frac{n}{p_j}$ ($j = 1, \dots, k$). Then (5) holds and so, using Corollary 2.5 (iv) \Rightarrow (i), we get an integer valued decomposition of f with smaller periods.

If the n -periodic function f can be written as a sum of periodic functions with smaller periods, then each term must be also periodic by a proper divisor of n , and then we can apply Corollary 2.5 (i) \Rightarrow (iv). □

3 A $\mathbb{Z} \times \mathbb{Z} \rightarrow \{0, 1\}$ example

Theorem 3.1 *There exists a function $u : \mathbb{Z} \times \mathbb{Z} \rightarrow \{0, 1\}$ that can be written as a sum of a $(0, 1)$ -periodic, a $(1, 0)$ -periodic and a $(1, 1)$ -periodic bounded $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ function, can be written also as the sum of three periodic $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ functions with the same periods, but cannot be written as a sum of three periodic bounded $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ functions with the same periods.*

Proof. Note first, that a function on $\mathbb{Z} \times \mathbb{Z}$ is $(0, 1)$ -periodic if and only if it is of the form $(x, y) \rightarrow f(x)$ for some function f on \mathbb{Z} , $(1, 0)$ -periodic iff it is of the form $(x, y) \rightarrow g(y)$, and $(1, 1)$ -periodic iff it is of the form $(x, y) \rightarrow h(x - y)$.

Fix a $t \in \mathbb{R} \setminus \mathbb{Q}$ and let

$$u(x, y) = [xt] - [yt] - [(x - y)t] \quad (x, y \in \mathbb{Z}), \quad (6)$$

where $[a]$ denotes the integer part of a . Using that $xt - yt - (x - y)t = 0$ we get

$$u(x, y) = -\{xt\} + \{yt\} + \{(x - y)t\}, \quad (7)$$

where $\{a\} = a - [a]$ is the fractional part of a . This shows that $u(x, y)$ is the sum of a $(0, 1)$ -periodic, a $(1, 0)$ -periodic and a $(1, 1)$ -periodic bounded $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ function. Expression (6) shows that $u(x, y) \in \mathbb{Z}$, and (7) shows that $-1 < u(x, y) < 2$, thus $u(x, y) \in \{0, 1\}$ for any x, y .

Suppose finally, that

$$u(x, y) = f(x) + g(y) + h(x - y). \quad (8)$$

We will show that f cannot be bounded and integer valued at the same time.

Given that $u(0, 0) = 0$, without any loss of generality we may assume that $f(0) = g(0) = h(0) = 0$. Substituting $x = y$ in (8) we get that $g(x) = -f(x)$ for every x , whereas the substitution $y = 0$ yields $h(x) = -f(x)$ for every x . Using these and substituting $y = 1$ in (8) we get

$$f(x) = f(x - 1) + u(x, 1) + f(1).$$

This implies that $f(1)$ determines f (and so g and h as well) everywhere. On the other hand, considering (8) as a function equation, (6) and (7) show that $f(x) = -g(x) = -h(x) = [xt]$ and $f(x) = -g(x) = -h(x) = -\{xt\}$ are both solutions of (8). Since $f(1)$, as a real number, can be written as $a[1 \cdot t] + (1 - a)(-\{1 \cdot t\})$ for some $a \in \mathbb{R}$, we get that every real solution of (8) is the affine linear combination of the solutions (6) and (7). In particular, f must be of the form

$$f(x) = a[tx] + (1 - a)(-\{tx\})$$

for some $a \in \mathbb{R}$. This shows that f is an integer valued function only if $a = 1$, and f is bounded only if $a = 0$. \square

Remark 3.2 One can check that the function u we constructed above can be also written as

$$u(x, y) = \begin{cases} 1, & \text{if } \{yt\} > \{xt\} \\ 0, & \text{if } \{yt\} \leq \{xt\}. \end{cases}$$

Corollary 3.3 *For any commutative group A that contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup there exist $a_1, a_2, a_3 \in A$ and a function $f : A \rightarrow \{0, 1\}$ such that f has a bounded real valued (a_1, a_2, a_3) -decomposition, but does not have a bounded integer valued (a_1, a_2, a_3) -decomposition. In particular, there exists $a_1, a_2, a_3 \in \mathbb{R}$ and a function $f : \mathbb{R} \rightarrow \{0, 1\}$ such that f has a bounded real valued (a_1, a_2, a_3) -decomposition, but does not have a bounded integer valued (a_1, a_2, a_3) -decomposition. \square*

Corollary 3.4 *If a commutative group A contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup, then the class of bounded $A \rightarrow \mathbb{Z}$ functions does not have the decomposition property. In particular, the class of bounded $\mathbb{R} \rightarrow \mathbb{Z}$ functions does not have the decomposition property. \square*

It is known (see e. g. in [3]) that a torsion-free commutative group contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ if and only if it is not isomorphic to any additive subgroup of \mathbb{Q} . Therefore combining Corollaries 2.4 and 3.4 we get the following characterization.

Corollary 3.5 *Let A be a torsion-free commutative group. Then the class of bounded $A \rightarrow \mathbb{Z}$ functions has the decomposition property if and only if it is isomorphic to an additive subgroup of \mathbb{Q} . \square*

4 Incommensurable periods and unprescribed periods

The following is an old unpublished result of the fourth author, which was rediscovered in [13, Theorem 2] by Mortola and Peirone, and independently in [2, Corollary 14] by Farkas and Révész: if $a_1, \dots, a_k \in \mathbb{R}$ are such that $a_i/a_j \notin \mathbb{Q}$ for any $i \neq j$, then a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has a real valued (a_1, \dots, a_n) -decomposition if and only if $\Delta_{a_1} \dots \Delta_{a_k} f = 0$. In fact, in [2, Theorem 15] a more general result is proved in a more general setting. The arguments in [13] and [2] work also for $f : \mathbb{R} \rightarrow \mathbb{Z}$ functions, and more generally for the following theorem. For completeness and for the readers' convenience we present a proof, similar to the proofs in [13] and [2]. By a multiple of an element g in a commutative group (that is, in a \mathbb{Z} -module) we mean any element of the form ng , where $n \in \mathbb{Z} \setminus \{0\}$.

Theorem 4.1 *Suppose that A and B are commutative groups, $a_1, \dots, a_k \in A$ are of infinite order, and no two of a_1, \dots, a_k have a common multiple. Then for any function $f : A \rightarrow B$ the following two statements are equivalent:*

- (i) *there exists a decomposition $f = f_1 + \dots + f_k$ such that each f_j is an a_j -periodic $A \rightarrow B$ function.*
- (ii) $\Delta_{a_1} \dots \Delta_{a_k} f = 0$.

Lemma 4.2 *(cf. [13, Lemma] and [2, Lemma 11]) Suppose that A and B are commutative groups, $a, b \in A$ are of infinite order and have no common multiple. Then for any b -periodic function $h : A \rightarrow B$ there exists a b -periodic function $g : A \rightarrow B$ such that $\Delta_a g = h$; or in other words, the operation Δ_a is onto on the class of b -periodic $A \rightarrow B$ functions.*

Proof. Define the relation \sim on A by $x \sim y \iff \exists m, n \in \mathbb{Z}, y - x = na + mb$. This is clearly an equivalence relation. Choose an element from each equivalence class. If x_0 is the chosen element from the equivalent class of an arbitrary $x \in A$, then $x = x_0 + na + mb$ for some $n, m \in \mathbb{Z}$; and since a and b are of infinite order having no common multiple, the pair n, m is unique. Hence the following definition gives a well defined function on A . Let

$$g(x_0 + na + mb) = \begin{cases} h(x_0) + h(x_0 + a) + \dots + h(x_0 + (n-1)a), & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -h(x_0 - a) - h(x_0 - 2a) - \dots - h(x_0 + na), & \text{if } n < 0. \end{cases}$$

Since $x + b = x_0 + na + (m+1)b$, we see that $g(x)$ and $g(x+b)$ are defined by the same formula, thus g is b -periodic. Using the definition of g and the b -periodicity of h we get for an arbitrary $x = x_0 + na + mb$ that

$$\begin{aligned} \Delta_a g(x) &= \Delta_a g(x_0 + na + mb) = g(x_0 + (n+1)a + mb) - g(x_0 + na + mb) \\ &= h(x_0 + na) = h(x_0 + na + mb) = h(x). \end{aligned}$$

Therefore $\Delta_a g = h$, which completes the proof. \square

Corollary 4.3 *If A and B are commutative groups, $a_1, \dots, a_k, b \in A$ are of infinite order and a_j and b have no common multiple for any j , then the operation $\Delta_{a_1} \dots \Delta_{a_k}$ is onto on the class of b -periodic $A \rightarrow B$ functions.* \square

Proof of Theorem 4.1. The implication (i) \Rightarrow (ii) is clear. We prove (ii) \Rightarrow (i) by induction. For $k=1$ the statement is clear. Suppose that (ii) \Rightarrow (i) holds for k , and now we prove it for $k+1$.

Assume that $\Delta_{a_1} \dots \Delta_{a_k} \Delta_{a_{k+1}} f = 0$, where $a_1, \dots, a_{k+1} \in A$ are of infinite order such that no two of them have a common multiple. Applying Corollary 4.3 for $b = a_{k+1}$ and $h = \Delta_{a_1} \dots \Delta_{a_k} f$ we get that there exists an a_{k+1} -periodic function $f_{k+1} : A \rightarrow B$ such that

$$\Delta_{a_1} \dots \Delta_{a_k} f_{k+1} = \Delta_{a_1} \dots \Delta_{a_k} f.$$

Finally, applying (ii) \Rightarrow (i) for k and for the function $f - f_{k+1}$ we get $f_1, \dots, f_k : A \rightarrow B$ such that each f_j is a_j -periodic and $f - f_{k+1} = f_1 + \dots + f_k$, so $f = f_1 + \dots + f_k + f_{k+1}$. \square

Corollary 4.4 *Let A, B be commutative groups, and let B' be a subgroup of B . Suppose that $a_1, \dots, a_k \in A$ are of infinite order and no two of them have a common multiple. If a function $f : A \rightarrow B'$ has a decomposition $f = f_1 + \dots + f_k$ such that each f_j is an a_j -periodic $A \rightarrow B$ function, then f also has a decomposition $f = g_1 + \dots + g_k$ such that each g_j is an a_j -periodic $A \rightarrow B'$ function.*

Proof. Note that (ii) of Theorem 4.1 is the same condition for B and B' . So we may apply first (i) \Rightarrow (ii) for B , and then (ii) \Rightarrow (i) for B' . \square

Although in Corollary 4.4 there are restrictions about the periods a_1, \dots, a_k , one can easily use this result for characterizing those $A \rightarrow B$ functions that can be written as a sum of periodic functions with any (unprescribed) periods, at least if A is torsion-free. (The following theorem is a generalization of a result of Mortola and Peirone [13, Theorem 3], who studied the special case when $A = B = \mathbb{R}$.)

Theorem 4.5 *For any torsion-free commutative group A , commutative group B and function $f : A \rightarrow B$, the following two statements are equivalent:*

- (i) *the function f can be written as the sum of finitely many periodic functions;*
- (ii) *there exist $a_1, \dots, a_k \in A$ such that no two of a_1, \dots, a_k have a common multiple and $\Delta_{a_1} \dots \Delta_{a_k} f = 0$.*

Proof. The implication (ii) \Rightarrow (i) immediately follows from Theorem 4.1. To prove (i) \Rightarrow (ii), suppose that $f = f_1 + \dots + f_n$ is a periodic decomposition, and let f_j be b_j -periodic ($j = 1, \dots, n$).

Let $\{a_1, \dots, a_k\} \subset A$ be a set with minimal cardinality such that for each b_j ($j = 1, \dots, n$) there exists an a_{i_j} ($i_j \in \{1, \dots, k\}$) such that a_{i_j} is a multiple of b_j . By minimality, no two of a_1, \dots, a_k have a common multiple. Letting

$$g_i = \sum_{j : i_j=i} f_j \quad (i = 1, \dots, k)$$

we get that $f = g_1 + \dots + g_k$, and that each g_i is a_i -periodic, which implies that $\Delta_{a_1} \dots \Delta_{a_k} f = 0$. \square

Corollary 4.6 *Let A be a torsion-free commutative group, let B be an arbitrary commutative group, and let B' be a subgroup of B . Then a function can be written as a finite sum of periodic $A \rightarrow B'$ functions if and only if it is an $A \rightarrow B'$ map that can be written as a finite sum of periodic $A \rightarrow B$ functions.*

Proof. This is clear from Theorem 4.5, since (ii) of Theorem 4.5 is the same condition for B and B' . \square

The special cases of Theorem 4.5 and Corollary 4.6 give the following characterizations of those functions that can be written as a sum of periodic integer valued functions.

Corollary 4.7 *For any integer valued function f on any torsion-free commutative group the following three statements are equivalent:*

- (i) *the function f can be written as the sum of finitely many integer valued periodic functions;*
- (ii) *the function f can be written as the sum of finitely many real valued periodic functions;*
- (iii) *there exist $a_1, \dots, a_k \in A$ such that no two of a_1, \dots, a_k have a common multiple and $\Delta_{a_1} \dots \Delta_{a_k} f = 0$.* \square

If we want to specify the periods as well, then we have a characterization only if the periods are commensurable or pairwise incommensurable.

Theorem 4.8 *Suppose that in a torsion-free commutative group A , nonzero elements a_1, \dots, a_n are given such that either they have a common multiple or no two of them have a common multiple. Then a function $f : A \rightarrow \mathbb{Z}$ has an integer valued (a_1, \dots, a_n) -decomposition if and only if it has a real valued (a_1, \dots, a_n) -decomposition.*

Proof. Assume first that $k_1 a_1 = \dots = k_n a_n = a$ holds with some nonzero integers k_1, \dots, k_n . Using the fact that every finitely generated torsion-free commutative group is of the form $\mathbb{Z} \times \dots \times \mathbb{Z}$ (see e.g. [3, Theorem 10.4]), one can easily check that there exists an element $b \in A$ such that $[k_1, \dots, k_n]b = a$. It follows that a_1, \dots, a_n belong to the cyclic subgroup of A generated by b , hence the subgroup A_0 generated by a_1, \dots, a_n must be isomorphic to \mathbb{Z} . Then we can get an integer valued decomposition of f on each coset $A_0 + t$ by Theorem 2.1.

If no two of $a_1, \dots, a_n \in A \setminus \{0\}$ have a common multiple, then Corollary 4.4 can be applied. \square

Corollary 4.9 *Suppose that $a_1, \dots, a_n \in \mathbb{R} \setminus \{0\}$ are commensurable or pairwise incommensurable. Then a function $f : \mathbb{R} \rightarrow \mathbb{Z}$ has an integer valued (a_1, \dots, a_n) -decomposition if and only if it has a real valued (a_1, \dots, a_n) -decomposition. \square*

5 Questions and remarks

We do not know whether we indeed need restrictions about the periods in Theorem 4.8 and Corollary 4.9.

Question 5.1 *Does the following statement hold?*

- (*) *If a function $f : \mathbb{R} \rightarrow \mathbb{Z}$ has a real valued (a_1, \dots, a_k) -decomposition for some $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{R}$, then f also has an integer valued (a_1, \dots, a_k) -decomposition.*

Question 5.2 *Does the following hold for every commutative group A ?*

- (**) *If a function $f : A \rightarrow \mathbb{Z}$ has a real valued (a_1, \dots, a_k) -decomposition for some $k \in \mathbb{N}$, $a_1, \dots, a_k \in A$, then f also has an integer valued (a_1, \dots, a_k) -decomposition.*

By Corollary 4.9, (*) holds for $k = 2$.

Note the following connections between Questions 5.1 and 5.2.

Proposition 5.3 *The following three statements are equivalent:*

- (i) (*) holds;
- (ii) (**) holds for every $A = \mathbb{Z} \times \dots \times \mathbb{Z}$;
- (iii) (**) holds for every torsion-free commutative group A .

Proof. Clearly (i) is a special case of (iii). It is also easy to see that (i) implies (ii), since $A = \mathbb{Z} \times \dots \times \mathbb{Z}$ can be always embedded in \mathbb{R} and then, by defining f to be zero on $\mathbb{R} \setminus A$, we can apply (*).

Therefore it is enough to prove (ii) \Rightarrow (iii). Using again that every finitely generated torsion-free commutative group is of the form $\mathbb{Z} \times \dots \times \mathbb{Z}$, we get that the additive subgroup of A generated by the periods a_1, \dots, a_k is of the form $\mathbb{Z} \times \dots \times \mathbb{Z}$. On the other hand, if we get a decomposition on this subgroup, the same decomposition works for each coset. \square

By Theorem 2.1, $(**)$ holds for $A = \mathbb{Z}$, which easily implies that $(**)$ holds for finite cyclic groups as well. But we do not know if $(**)$ holds for $A = \mathbb{Z} \times \mathbb{Z}$, and whether it holds for every finite commutative group, not even for groups of form $\mathbb{Z}_n \times \mathbb{Z}_n$.

We remark that for a finite commutative group A , $(**)$ is equivalent with the statement that the class of all $A \rightarrow \mathbb{Z}$ functions has the decomposition property. Indeed, $A \rightarrow \mathbb{R}$ functions are always bounded for any finite set A , and after Laczkovich and Révész [12] we know that the class of bounded $A \rightarrow \mathbb{R}$ functions has the decomposition property, so the same argument would work as in the proof of Corollary 2.2.

We do not know whether we indeed need restrictions about the periods in Corollary 4.4. We might have a positive answer even for the following most general question.

Question 5.4 *Does the following statement hold for arbitrary commutative groups A, B and B' such that $B' \leq B$?*

$(***)$ *If a function $f : A \rightarrow B'$ has a B -valued (a_1, \dots, a_k) -decomposition for some $k \in \mathbb{N}$, $a_1, \dots, a_k \in A$, then f also has a B' -valued (a_1, \dots, a_k) -decomposition.*

References

- [1] T.M. Apostol, *Resultants of cyclotomic polynomials*, Proc. Amer. Math. Soc. **24** (1970), 457–462.
- [2] B. Farkas, Sz.Gy. Révész, *Decomposition as the sum of invariant functions with respect to commuting transformations*, Aequationes Math. **73** (2007), 233–248.
- [3] L. Fuchs, *Abelian Groups*, Akadémiai Kiadó, Budapest, 1958, and Pergamon Press, New York, 1960.
- [4] Z. Gajda, *Note on decomposition of bounded functions into the sum of periodic terms*, Acta Math. Hungar. **59** (1992), 103–106.
- [5] V.M. Kadets, S.B. Shumyatskiy, *Averaging technique in the periodic decomposition problem*, Mat. Fiz. Anal. Geom. **7** (2000), 184–195.
- [6] V.M. Kadets, S.B. Shumyatskiy, *Additions to the periodic decomposition theorem*, Acta Math. Hungar. **90** (2001), 293–305.
- [7] T. Keleti, *On the differences and sums of periodic measurable functions*, Acta Math. Hungar. **75** (1997), 279–286.
- [8] T. Keleti, *Difference functions of periodic measurable functions*, Fund. Math. **157** (1998), 15–32.
- [9] T. Keleti, *Periodic decomposition of measurable integer valued functions*, manuscript.
- [10] T. Keleti, M. Kolountzakis, *On the determination of sets by their triple correlation in finite cyclic groups*, Online J. Anal. Combin. **1** (2006), #4.

- [11] M. Laczkovich, Sz.Gy. Révész, *Periodic decompositions of continuous functions*, Acta Math. Hungar. **54** (1989), 329–341.
- [12] M. Laczkovich, Sz.Gy. Révész, *Decompositions into periodic functions belonging to a given Banach space*, Acta Math. Hungar. **55** (1990), 353–363.
- [13] S. Mortola, R. Peirone, *The sum of periodic functions*, Boll. Un. Mat. Ital. B(8) **2** (1999), 393–396.
- [14] T. Natkaniec, W. Wilczyński, *Sums of periodic Darboux functions and measurability*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 369–376.
- [15] M. Wierdl, *Continuous functions that can be represented as the sum of finitely many periodic functions*, Mat. Lapok **32** (1984), 107–113 (in Hungarian).
- [16] B.L. van der Waerden, *Algebra I, Neunte Auflage*, Springer, 1993.

DEPARTMENT OF ALGEBRA AND NUMBER THEORY
 EÖTVÖS LORÁND UNIVERSITY
 PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117 BUDAPEST, HUNGARY
Email address: karolyi@cs.elte.hu
URL: <http://www.cs.elte.hu/~karolyi>

DEPARTMENT OF ANALYSIS
 EÖTVÖS LORÁND UNIVERSITY
 PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117 BUDAPEST, HUNGARY
Email address: elek@cs.elte.hu
URL: <http://www.cs.elte.hu/anal/keleti>

DEPARTMENT OF ANALYSIS
 EÖTVÖS LORÁND UNIVERSITY
 PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117 BUDAPEST, HUNGARY
 AND
 COMPUTER AND AUTOMATION RESEARCH INSTITUTE
 KENDE U. 13-17, H-1111 BUDAPEST, HUNGARY
Email address: kosgeza@cs.elte.hu

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
 HUNGARIAN ACADEMY OF SCIENCES
 P.O. BOX 127, H-1364 BUDAPEST, HUNGARY
Email address: ruzsa@renyi.hu