The balls do not generate all Borel sets using complements and countable disjoint unions

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Abstract

We prove that in a separable infinite dimensional Hilbert space the Dynkin system generated by the family of all open balls (that is, the smallest collection containing the open balls and closed under complements and countable disjoint unions) does not contain all Borel sets.

A non-empty family $\mathcal{D}$ of subsets of a set X is said to be a *Dynkin system* provided $\mathcal{D}$ is closed under complements and countable disjoint unions. (The terms 'σ-class', 'concrete quantum logic' and 'q-σ-algebra' is also used for the same notion; the latter two in the study of quantum logic.) If X is a metric space then the Dynkin system $\mathcal{D}(X)$ generated by the family of all balls of X (that is, the smallest Dynkin system that contains all balls of X) is clearly a subclass of the family $\mathcal{B}(X)$ of the Borel subsets of X. (Although it doesn't make much difference, by a ball we always mean here an open ball.)

If two Borel probability measures agree on each ball then they clearly agree on the sets of $\mathcal{D}(X)$, so if we knew that $\mathcal{D}(X) = \mathcal{B}(X)$ then we could deduce that the two measures must be the same. This was one of the motivations of studying the question whether $\mathcal{D}(X) = \mathcal{B}(X)$.

In 1971 Davies [1] constructed a compact metric space P with two different Borel probability measures on P agreeing on every ball, so neither the above determination statement nor $\mathcal{D}(X) = \mathcal{B}(X)$ can be true in general. Later these questions were investigated in separable Banach spaces. The question

($*$) *Is $\mathcal{D}(X) = \mathcal{B}(X)$ true for any separable Banach space X?*

arose for example at the 3rd Conference of Topology and Measure held in Vitte and Hiddensee in 1980, as a possible way for answering the question

($**$) *Is it true in any separable Banach space that two different finite Borel measures cannot agree on every ball?*

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In 1991 Preiss and Tišer [9] gave an affirmative answer to $(**)$ using an approach not based on $(*)$. In particular, the question $(*)$ remained open.

Olejček ([6], [7]) proved $\mathcal{D}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R}^n)$ for $n = 2$ in 1988 and for $n = 3$ in 1995; and recently Jackson and Mauldin [2] and independently Zelený [12] proved that $\mathcal{D}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R}^n)$ for any positive integer n. In fact, in both papers slightly more is proved: Jackson and Mauldin proved $\mathcal{D}(X) = \mathcal{B}(X)$ for any finite dimensional Banach space; Zelený used weaker operations (countable monotone unions and intersections and countable disjoint unions).

Another positive result about $(*)$ is E. Riss's Theorem: Any separable Banach space can be renormed in such a way that every Borel set belongs to the Dynkin system generated by the new balls. Moreover, it is enough to use balls with radii > 1. (In this form, this result can be found in [5]; the original ideas are in [10] and [11].)

In this paper we give a negative answer to $(*)$, which shows that one cannot prove the positive answer of Preiss and Tišer to $(**)$ via $(*)$. More precisely, we show

Theorem 1 *The Dynkin system generated by the family of all balls in a separable infinite dimensional Hilbert space does not contain all Borel sets.*

The method of the proof can be extended to somewhat more general situation; its natural assumptions seem to be that the norm is uniformly smooth and uniformly convex. However, some special features of the geometry of balls in Hilbert spaces are used; in particular, the argument used to show Lemma 9 would need substantial modifications. We preferred the simplicity to generality; in particular, since, by the above mentioned theorem of E. Riss's we cannot get the same negative result in a very general context: the answer must depend on the norm.

Another extension of the result presented here can be to finite dimensional Hilbert spaces. We know from the above mentioned results of Jackson, Mauldin and Zelený that the statement of Theorem 1 does not hold in finite dimension; however, the proof presented here carries over without any essential change to showing that, in Euclidean spaces of dimension at least three, the Dynkin system generated by the family of all balls with radii > 1 does not contain all Borel sets. (One just replaces 'cap null sets' in the argument below by countable sets, and uses that the first set from Lemma 4 is countable; the second set of this Lemma is not used if only large balls are considered. See [3] for the many simplifications that can be made in this case.) But this remark is highly misleading: The finite dimensional result was not obtained as a generalisation of the infinite dimensional one; the finite dimensional result of [3] was proved first and the infinite dimensional one was obtained by a generalisation of its proof. However, the motivation for [3] came from infinite dimensions: The finite dimensional problem was posed because of the following strange similarity between small balls in an infinite dimensional Hilbert space and large balls in the finite dimensional one:

On a Hilbert space H;

(1) *"$\mu(B) \geq \nu(B)$ for every ball B with radius < 1 implies $\mu \geq \nu$" holds if and only if $dim(H) < \infty$.*

(2) *"$\mu(B) \geq \nu(B)$ for every ball B with radius > 1 implies $\mu \geq \nu$" holds if and only if $dim(H) = \infty$.*

(These results can be found in [4] but the proof of the main statement is in [8]. For a more accessible proof see [5]).

The analogy, although at that time not well founded, turned out to be true enough to enable the solution of the infinite dimensional question. In this world of strange analogies, it is interesting to observe that the earlier mentioned theorem of E. Riss provides an equivalent norm on a Hilbert space such that the Dynkin system generated by large balls coincides with the Borel sets, and that we do not know if this is also true for small balls.

The idea of the proof of Theorem 1 is to show that the sets from the Dynkin class are, modulo a certain σ-ideal of "small" sets, obtained by a relatively simple operation directly from balls. To simplify some geometric considerations, we consider, by means of the stereographic projection, a subset of the given infinite dimensional separable Hilbert space H' as a subset of the unit sphere $X = \{x \in H : \|x\| = 1\}$ of $H = H' \times \mathbf{R}$. Since in this identification balls in H' become caps in X, where a *cap* in X is defined as a set of the form $C(y,r) = \{x \in X : \langle x, y\rangle > 1 - r\}$, where $y \in X$ and $r \in \mathbf{R}$, it suffices to show that the Dynkin system generated by caps in X does not contain all Borel sets.

We first collect a few simple facts concerning the geometry of spheres in X. Under a *sphere* we will understand any set of the form $S(y,a) = \{x \in X : \langle x, y\rangle = a\}$, where $y \in X$ and $|a| < 1$. Note that the boundary of any cap C is either empty (if $C = X$ or $C = \emptyset$), contains exactly one point x (if $C = X \setminus \{x\}$), or it is a sphere (in which case we call the cap C proper). We equip the set of spheres by the Hausdorff distance defined by $d(S,T) = \max(\sup_{x\in S} d(x,T), \sup_{y\in T} d(y,S))$. We will need very little about this metric; most of it is contained in the following Lemma, in which we call a non-empty family of sets monotone if for every pair U, V of sets belonging to it either $U \subset V$ or $V \subset U$. In particular, it is not important if we use the metric induced by the norm or if we use the geodesic distance.

Lemma 2 *Suppose that $\mathcal{C}$ is a monotone family of caps in X. Then the union of all elements of $\mathcal{C}$ is a cap. Moreover, if the union is a proper cap then its boundary belongs to the closure of the set of the boundaries of the caps from $\mathcal{C}$.*

Proof. We may assume that each $C(y,r) \in \mathcal{C}$ has $0 < r < 2$. Let s be the supremum of all possible r and choose $0 < r_k \leq s$ and $y_k \in X$ such that $r_k \to s$ and $C(y_k, r_k) \in \mathcal{C}$. If $s = 2$, then $X \setminus C(y_k, r_k)$ is a decreasing sequence of closed sets with diameter tending to zero; letting $-y$ be the only point of its intersection, we see from the monotonicity of $\mathcal{C}$ that its union is $C(y, 2)$. If

$s < 2$, we have $||y_k - y_l|| \leq c|r_k - r_l|$, where c is a constant. Hence the sequence y_k converges to some $y \in X$ and the monotonicity of $\mathcal{C}$ implies that its union is $C(y, s)$; moreover, since $y_k \to y$ and $r_k \to r$, the boundaries of $C(y_k, r_k)$ converge to the boundary of $C(y, r)$, which proves the last statement of the Lemma. □

Two spheres S, T may be either equal, in which case $X \setminus (S \cup T)$ has exactly two components, or their intersection may contain exactly one point in which case $X \setminus (S \cup T)$ has exactly three components, or their intersection is an infinite set, in which case $X \setminus (S \cup T)$ has exactly four components. In the last case we shall say that S, T are *crossing*.

The intersection of a finite number of spheres is always a set of the form $W \cap X$, where W is a closed affine subspace of H of finite co-dimension.

The points $x, y \in X$ are said to be *separated* by a sphere S if they lie in different components of $X \setminus S$.

Lemma 3 *Let $S \neq T$ be crossing spheres. Then there is $\varepsilon > 0$ such that any two spheres S', T' with $d(S', S) < \varepsilon$ and $d(T', T) < \varepsilon$ are crossing.*

PROOF. Choose four points u_1, u_2, u_3, u_4, one from each of the four components of $X \setminus (S \cup T)$. Each pair from u_1, u_2, u_3, u_4 is separated by either S or T, so, if $\varepsilon > 0$ is small enough, it is separated by either S' or T'. Hence u_1, u_2, u_3 and u_4 are in different components of $H \setminus (S' \cup T')$, which gives that S', T' are crossing. □

For our purpose, a set is 'negligible' if it belongs to the σ-ideal of the subsets of X generated by those sets N whose every point is contained in an arbitrarily small cap C such that $N \cap \partial C = \emptyset$. The sets from this σ-ideal will be termed *cap null*. We need some simple facts about these null sets.

Lemma 4 *Suppose that $\mathcal{S}$ is a family of pairwise non-crossing spheres. Then the following sets are cap null.*

(1) *The set N^1 of those points that belong to at least two different spheres from $\mathcal{S}$.*

(2) *The set N^2 of points x that are contained in arbitrarily small caps whose boundary belongs to $\mathcal{S}$.*

PROOF. We show that N^1 is, in fact, countable. Let M be a dense countable subset of X. For every $x \in N^1$ choose spheres $S, T \in \mathcal{S}$ such that $S \neq T$ and $x \in S \cap T$; denote by U, V, W the three components of $X \setminus (S \cup T)$, where the notation is such that $\partial U = S$, $\partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, and choose points $u \in M \cap U$, $v \in M \cap V$, and $w \in M \cap W$. We claim that x is uniquely determined by the triple $\langle u, v, w \rangle$; since the set of such triples is countable, this will finish the proof of (1). To prove the claim, suppose, that from some $x' \in N^1$, $x' \neq x$, we arrived to the same $\langle u, v, w \rangle$ using spheres $S', T' \in \mathcal{S}$ and components U', V', W' of $X \setminus (S' \cup T')$. Since the spheres from

$\mathcal{S}$ are non-crossing and since $U \cap U' \neq \emptyset$, we have that $U \subset U'$ or $U' \subset U$; similarly for V's and W's. Exchanging the role of x and x' and/or of U's and V's if necessary, there are only two cases to consider: (a) $U \supset U'$ and $V \supset V'$ and (b) $U \subset U'$, $V \supset V'$ and $W \subset W'$. In case (a) we recall that $\overline{U} \cap \overline{V}$ contains only x and that $x' \in \overline{U'} \cap \overline{V'}$, so $x = x'$. In case (b) we get from $W \subset W'$ that $U' \subset \overline{U \cup V}$; so since U' is open and connected, and $\overline{U} \cap \overline{V}$ is just one point, we infer that $U' = U$ and we are back in the already proved case (a).

To prove (2), we just need to show that $N^2 \cap S = \emptyset$ for every $S \in \mathcal{S}$. But this is obvious, since for any $x \in S$ the boundary of every sufficiently small open cap containing x crosses S, so it cannot belong to $\mathcal{S}$. □

Lemma 5 *Suppose that $\mathcal{S}$ is a non-empty closed family of pairwise non-crossing spheres and that $y \in X \setminus \bigcup_{S \in \mathcal{S}} S$ is not contained in caps of arbitrarily small diameter whose boundary is in $\mathcal{S}$. Then there is a (finite or infinite) sequence of pairwise disjoint caps $C_1, C_2, \ldots$ such that $\partial C_i \in \mathcal{S}$ for all i, $y \notin \bigcup_k \overline{C_k}$, and no point of $X \setminus \bigcup_k \overline{C_k}$ is separated from y by a sphere from $\mathcal{S}$.*

Proof. Let $\mathcal{C}$ be the family of all caps C with boundary in $\mathcal{S}$ such that $y \notin C$. Since the spheres from $\mathcal{S}$ are non-crossing, for every $C, C' \in \mathcal{C}$ we have that $C \cap C' = \emptyset$ or $C \subset C'$ or $C' \subset C$. According to Lemma 2 and Hausdorff maximal principle, each cap from $\mathcal{C}$ is a subset of a maximal cap from $\mathcal{C}$; these maximal caps are disjoint, hence there are only countably many of them and we may order them into a sequence $C_1, C_2, \ldots$. By definition, every point of X that is separated from y belongs to some cap from $\mathcal{C}$, hence to some C_k. □

Lemma 6 *There is $s_0 < 1$ such that no three caps $C(u_i, r_i)$ with $r_i \geq s_0$ are pairwise disjoint.*

Proof. The centres of the caps $C(u, r)$ that are disjoint from $C(u_1, s_0)$ and satisfy $r \geq s_0$ belong to a cap $C(-u, \varepsilon)$, where $\varepsilon > 0$ is small if s_0 is close to one. So u_1, u_2 have to belong to $C(-u, \varepsilon)$, and the caps $C(u_2, r_2)$ and $C(u_3, r_3)$ are not disjoint. □

Lemma 7 *Suppose that $C_1, C_2, \ldots$ is a sequence of caps which can be partitioned into two pairwise disjoint sequences and has the property that no union $\overline{C_i} \cup \overline{C_j}$ covers X. Then there are $x \in X \setminus \overline{C_1}$ and $\varepsilon > 0$ such that no union $\overline{C_i} \cup \overline{C_j}$ can contain the boundary of any cap $C(x, r)$ for $0 < r < \varepsilon$.*

Proof. Since every cap from the sequence $C_1, C_2, \ldots$ may be contained in no more than one other cap, we may achieve that no C_i is contained in another C_j; using that no union $\overline{C_i} \cup \overline{C_j}$ covers X we infer that $\partial C_i \setminus C_j \neq \emptyset$ for each $i \neq j$.

If there are $1 < i < j$ such that ∂C_i and ∂C_j are crossing, choose $x \in \partial C_i \cap \partial C_j$. Then for sufficiently small $r > 0$ the boundary of $C(x, r)$ meets all four sets $C_i \cap C_j$, $C_i \setminus \overline{C_j}$, $C_j \setminus \overline{C_i}$ and $X \setminus (\overline{C_i} \cup \overline{C_j})$. Note that no union $\overline{C_m} \cup \overline{C_n}$ with $C_m \cap C_n = \emptyset$ can contain the boundary of $C(x, r)$, since $x \notin C_m \cup C_n$ and $\overline{C_m} \cup \overline{C_n} \neq X$. Using that the sequence $C_0, C_1, \ldots$ may be partitioned into two pairwise disjoint sequences, we see that the only ways in which $\overline{C_m} \cup \overline{C_n}$ could cover $\partial C(x,r)$ are (up to symmetry): (1) $C_m = C_i$ and $C_n = C_j$; but then the union $\overline{C_m} \cup \overline{C_n}$ does not meet $X \setminus (\overline{C_i} \cup \overline{C_j})$. (2) $C_m = C_i$ and $C_n \cap C_j = \emptyset$, but then $\overline{C_m} \cup \overline{C_n}$ does not meet $C_j \setminus \overline{C_i}$. (3) $C_m \cap C_i = \emptyset$ and $C_n \cap C_j = \emptyset$, but then $\overline{C_m} \cup \overline{C_n}$ does not meet $C_i \cap C_j$.

If ∂C_i and ∂C_j are non-crossing for $1 < i < j$ and the sequence $C_1, C_2, \ldots$ has at least two terms, we choose any $x \in \partial C_2 \setminus \overline{C_1}$. If $r > 0$ is so small that $\overline{C(x,r)} \cap \overline{C_1} = \emptyset$, then the only way to cover $\partial C(x,r)$ by $\overline{C_i} \cup \overline{C_j}$ is to have $i = 2 < j$. But then $C_i \cap C_j = \emptyset$ would imply that $\overline{C_i} \cup \overline{C_j} = X$.

Finally, if the sequence contains only C_1, we choose any $x \in X \setminus \overline{C_1}$, which is non-empty according to the assumptions. □

Lemma 8 *Suppose that $C_1, C_2, \ldots$ is a sequence of caps which can be partitioned into two pairwise disjoint sequences and has the property that no union $\overline{C_i} \cup \overline{C_j}$ covers X. Then $X \not\subset \bigcup_i \overline{C_i}$ and for every $x \in X \setminus \bigcup_i \overline{C_i}$ there is $\varepsilon > 0$ such that no union $\overline{C_i} \cup \overline{C_j}$ covers the boundary of any cap C such that $x \in C$ and* $\mathrm{diam}(C) < \varepsilon$.

Proof. By Lemma 7 there are $x_1 \in X \setminus \overline{C_1}$ and $0 < \varepsilon_1 < \mathrm{dist}(x_1, \overline{C_1})$ such that no union $\overline{C_i} \cup \overline{C_j}$ can contain the boundary of the cap $C(x_1, \varepsilon_1)$. Choose an affine subspace W_1 of H such that $\partial C(x_1, \varepsilon_1) = X \cap W_1$ and use Lemma 7 in the space $X \cap W_1$ to find $x_2 \in X \setminus \overline{C_2}$ and $0 < \varepsilon_2 < \mathrm{dist}(x_2, \overline{C_2})/2$ such that no union $\overline{C_i} \cup \overline{C_j}$ can contain the boundary of the cap $C(x_2, \varepsilon_2)$. Choose an affine subspace W_2 of H such that $X \cap W_1 \cap \partial C(x_2, \varepsilon_2) = X \cap W_2$, etc. Continuing in this way, we construct a convergent sequence x_k; its limit clearly does not belong to $\bigcup_i \overline{C_i}$. Hence $X \not\subset \bigcup_i \overline{C_i}$.

To prove the second statement, note that (by Lemma 6) there are at most four caps among $C_k = C(u_k, r_k)$ that satisfy $r_k \geq s_0$. Let $x \in X \setminus \bigcup_i \overline{C_i}$. Choose $\varepsilon > 0$ so small that for every U containing x and having diameter $< \varepsilon$ we have that $\overline{U} \cap \overline{C_k} = \emptyset$ for each of these large caps and that every closed cap $\overline{C}(v,s)$ meeting $\overline{U}$ and containing $-x$ has $s > s_0$. If the boundary of such a U is covered by $\overline{C_i} \cup \overline{C_j}$ then both $\overline{C_i}$ and $\overline{C_j}$ must meet $\overline{U}$, hence both x and $-x$ belong to $V = X \setminus (\overline{C_i} \cup \overline{C_j})$, so V is disconnected, which is a contradiction. □

Lemma 9 *Suppose that W is an infinite dimensional closed affine subspace of H meeting X in more than one point and that $C_1, C_2, \ldots$ is a sequence of caps which can be partitioned into two pairwise disjoint sequences and has the property that no union $\overline{C_i} \cup \overline{C_j}$ covers $X \cap W$. Then the set $(X \cap W) \setminus \bigcup_i \overline{C_i}$ is not cap null.*

Proof. Suppose that $(X \cap W) \setminus \bigcup_i \overline{C_i} \subset \bigcup_{k=1}^{\infty} N_k$, where $N_k \subset X$ are such that every $x \in N_k$ is contained in an arbitrarily small cap C such that $N_k \cap \partial C = \emptyset$. We let $W_1 = W$. By induction we shall define $W_1 \supset W_2 \supset \ldots$ such that each W_p has the properties we required for W in the conditions of the lemma. Assuming that W_p has been defined, we can apply Lemma 8 for the space $X \cap W_p$ instead of X: Let k_p be the least index for which there is $x_p \in (X \cap W_p \setminus \bigcup_i \overline{C}_i) \cap N_{k_p}$ and let ε_p be such that $0 < \varepsilon_p < \mathrm{dist}(x_p, C_p)/2^p$ and no union $\overline{C_i} \cup \overline{C_j}$ covers any set of the form $X \cap W_p \cap \partial C$, where C is a cap such that $x_p \in C$ and $\mathrm{diam}(C) < \varepsilon_p$.

Let D_p be a cap with $\mathrm{diam}(D_p) < \varepsilon_p$ such that $x_p \in D_p$ and $N_{k_p} \cap \partial D_p = \emptyset$ and let W_{p+1} be an affine subspace of W_p such that $X \cap W_{p+1} = X \cap W_p \cap \partial D_p$. In this way we obtain a decreasing sequence $X \cap W_{p+1}$ of closed sets with diameter tending to zero; this sequence has a non-empty intersection; but the (unique) point x of this intersection satisfies $x \in (X \cap W) \setminus \bigcup_i \overline{C_i}$ and $x \notin \bigcup_{k=1}^{\infty} N_k$. □

We now introduce a purely technical notion of a type of subsets of X, which we will term K-sets. As we will see in Lemma 12, every set from the Dynkin class generated by the caps is in certain sense composed from these sets. A set K will be called a *K-set about a sphere S* if either $K = S$ or it is a non-empty set of the form $K = X \setminus \bigcup_j \overline{C_j}$, where C_j is a (finite or infinite) sequence of pairwise disjoint caps and $\partial C_0 = S$.

Lemma 10 *Suppose that K and K' are K-sets about crossing spheres S and S', respectively. Then $K \cap K'$ is not cap null.*

PROOF. If $K = S$ and $K' = S$, use Lemma 9 with any C_i and with W chosen so that $X \cap W = S \cap S'$.

If $K = X \setminus \bigcup_i \overline{C_i}$, where C_i are disjoint caps, $S = \partial C_0$, and if $K' = S'$, Lemma 9, used with a W such that $S' = X \cap W$, shows that the only way in which $K \cap K'$ is not cap null is when some $\overline{C_i} \cup \overline{C_j} \supset S'$. Since C_i and C_j are disjoint this would imply that either $S' \subset \overline{C_i}$, which is impossible since $S = \partial C_0$ and S, S' are crossing, or $S' \subset \overline{C_j}$, which is impossible for the same reason, or $\overline{C_i} \cup \overline{C_j} = X$, which is impossible since it would mean that $K = \emptyset$.

Finally, if $K = X \setminus \bigcup_i \overline{C_i}$, where C_i are disjoint caps, $S = \partial C_0$, $K' = X \setminus \bigcup_i \overline{C_i'}$, where C_i' are disjoint caps, $S' = \partial C_0'$, then Lemma 9 with $W = H$ shows that the only possibility that we have to exclude is that $\overline{C_k} \cup \overline{C_l'} = X$ for some k, l. (Here we used again that K-sets are non-empty, so the union of the closures of two caps from the same family cannot cover X.) If this happens, let $D = X \setminus \overline{C_k}$ and $D' = X \setminus \overline{C_l'}$. Then either $k = 0$ and $S = \partial C_0 \subset \overline{D}$ or $k > 0$ and C_0, being disjoint from C_k, is a subset of D, so again $S \subset \overline{D}$. Similarly we see that $S' \subset \overline{D'}$. But D and D' are disjoint caps, so S and S' are not crossing. □

Lemma 11 *If K is a K-set about a sphere S, and G is an open subset of X meeting S, then $G \cap K$ is not cap null.*

PROOF. Take any cap C' containing a point of $G \cap S$ such that $\overline{C'} \subset G$ and use Lemma 10 with $K' = S' = \partial C'$. □

Lemma 12 *For every set E from the Dynkin system generated by the caps one may find a cap null set N and a family $\mathcal{S}$ of spheres satisfying*

(1) *for every $S \in \mathcal{S}$ and $\varepsilon > 0$ there is a sphere S' such that $d(S', S) < \varepsilon$ and for some K-set K' about S' one of the following two alternatives occurs:*

   (a) *$S \setminus N \subset E^c$ and $K' \setminus N \subset E$, or*

   (b) *$S \setminus N \subset E$ and $K' \setminus N \subset E^c$.*

(2) *Every pair of points* $x \in E \setminus (N \cup \bigcup_{S\in\mathcal{S}} S)$ *and* $y \in E^c \setminus (N \cup \bigcup_{S\in\mathcal{S}} S)$ *is separated by some* $S \in \mathcal{S}$.

PROOF. If E is a proper cap, it suffices to take $\mathcal{S} = \{\partial E\}$; for (1) one notes that $S' = S$ and $K' = E$ satisfy the first alternative, and (2) is obvious. If E has the properties (1) and (2) with some N and $\mathcal{S}$, then $X \setminus E$ has these properties with the same N and $\mathcal{S}$. Hence we just have to find the required set N and family $\mathcal{S}$ assuming that $E = \bigcup_{k=1}^{\infty} E_k$, the sets E_k are disjoint, and there are cap null sets N_k and families $\mathcal{S}_k$ such that for each k the following two conditions holds.

(1_k) for every $S \in \mathcal{S}_k$ and $\varepsilon > 0$ there is a sphere S' such that $d(S', S) < \varepsilon$ and for some K-set K' about S' one of the following two alternatives occurs:

- (a) $S \setminus N_k \subset E_k^c$ and $K' \setminus N_k \subset E_k$, or
- (b) $S \setminus N_k \subset E_k$ and $K' \setminus N_k \subset E_k^c$.

(2_k) Every pair of points $x \in E_k \setminus (N_k \cup \bigcup_{S\in\mathcal{S}_k} S)$ and $y \in E_k^c \setminus (N_k \cup \bigcup_{S\in\mathcal{S}_k} S)$ is separated by some $S \in \mathcal{S}_k$.

Since a sphere S is also a K-set about S, (1_k) implies that

($1'_k$) for every $S \in \mathcal{S}_k$ and $\varepsilon > 0$ there are K-sets K', K'' about spheres S', S'', respectively, such that $d(S', S) < \varepsilon$, $d(S'', S) < \varepsilon$, $K' \setminus N_k \subset E_k$ and $K'' \setminus N_k \subset E_k^c$.

We claim that any $S \in \mathcal{S}_m$ and $T \in \mathcal{S}_n$ are non-crossing. Indeed, suppose that S and T are crossing. Use Lemma 3 to find $\varepsilon > 0$ such that any two spheres S', T' satisfying $d(S', S) < \varepsilon$ and $d(T', T) < \varepsilon$ are crossing. If $m = n$, we use ($1'_m$) to find K-sets K', L' about spheres S', T', respectively, such that $d(S', S) < \varepsilon$, $d(T', T) < \varepsilon$, $K' \setminus N_m \subset E_m$ and $L' \setminus N_m \subset E_m^c$. So S', T' are crossing and $K' \cap L' \subset N_m$, which contradicts Lemma 10. If $m \neq n$, we use ($1'_m$) and ($1'_n$) to find K-sets K', L' about spheres S', T', respectively, such that $d(S', S) < \varepsilon$, $d(T', T) < \varepsilon$, $K' \setminus N_m \subset E_m$ and $L' \setminus N_n \subset E_n$. So S', T' are crossing and $K' \cap L' \subset N_m \cup N_n$ (here is where we use that E_m and E_n are disjoint), which contradicts Lemma 10.

Define $\mathcal{S}'$ as the family of those spheres that belong to at least one of the $\mathcal{S}_k$ and $\mathcal{S}''$ as the closure of $\mathcal{S}'$ (in the metric d). According to the above claim, Lemma 3 implies that any $S, T \in \mathcal{S}''$ are non-crossing. Let N_0 be the set of those points that either belong to at least two different spheres from $\mathcal{S}''$ or are contained in arbitrarily small open caps whose boundary belongs to $\mathcal{S}''$. By Lemma 4, N_0 is cap null, and so $N = \bigcup_{k=0}^{\infty} N_k$ is cap null.

We show that every sphere $T \in \mathcal{S}''$ satisfies $T \setminus N \subset E$ or $T \setminus N \subset E^c$. Indeed, suppose to the contrary that $x \in (T \cap E) \setminus N$ and $y \in (T \setminus E) \setminus N$, and choose k such that $x \in E_k$. Since $x \in E_k \setminus N_k$ and $y \in E_k^c \setminus N_k$, we infer from ($1_k$) that $T \notin \mathcal{S}_k$. By definition of N_0, since $x \in T$, no other sphere from $\mathcal{S}'' \supset \mathcal{S}_k$ may contain x; and similarly for y. Hence $x \in E_k \setminus (N_k \cup \bigcup_{S\in\mathcal{S}_k} S)$ and $y \in E_k^c \setminus (N_k \cup \bigcup_{S\in\mathcal{S}_k} S)$, so ($2_k$) implies that x, y are separated by some

$S \in \mathcal{S}_k$. But then S and T are crossing, which contradicts the claim proved above.

Finally, we define $\mathcal{S}$ as the family of those spheres $S \in \mathcal{S}''$ that satisfy (1). We prove that every $S \in \mathcal{S}''$ such that $S \cap (E^c \setminus N) \neq \emptyset$ belongs to $\mathcal{S}$: The preceding paragraph shows that then $S \setminus N \subset E^c$. Moreover, for every $\varepsilon > 0$ we find $T \in \mathcal{S}_k$ such that $d(T,S) < \varepsilon/2$ and use $(1'_k)$ to find a K-set K' about a sphere S' such that $d(S',T) < \varepsilon/2$ and $K' \setminus N_k \subset E_k$; hence $K' \setminus N \subset K' \setminus N_k \subset E_k \subset E$, and we have proved the first alternative of (1).

It remains to show that $\mathcal{S}$ satisfies (2). Let $x \in E \setminus (N \cup \bigcup_{S \in \mathcal{S}} S)$ and $y \in E^c \setminus (N \cup \bigcup_{S \in \mathcal{S}} S)$. Then $\mathcal{S}''$ is non-empty, since some sphere from the $\mathcal{S}_k$ for which $x \in E_k$ must separate x and y. If $T \cap (E^c \setminus N) \neq \emptyset$ for some $T \in \mathcal{S}''$ that separates x and y, then (by the preceding paragraph) this T belongs to $\mathcal{S}$, and we are done. We may therefore assume that $S \setminus N \subset E$ for every $S \in \mathcal{S}''$ that separates x and y. From the definition of N_0 we infer that y is not contained in caps of arbitrarily small diameter whose boundary is in $\mathcal{S}''$. Hence Lemma 5 provides us with a sequence $C_0, C_1, \ldots$ of pairwise disjoint caps such that $\partial C_i \in \mathcal{S}''$, $y \notin \bigcup_k \overline{C_k}$, and no point of $X \setminus \bigcup_k \overline{C_k}$ is separated from y by a sphere from $\mathcal{S}''$. Since x is separated from y by a sphere from $\mathcal{S}''$, and does not belong to the boundary of any of the sets C_i, it is in some C_i, which we may rename as C_0. Let $S = \partial C_0$ and $K = X \setminus \bigcup_k \overline{C_k}$. Since S separates x and y, we have $S \setminus N \subset E$. Finally, since every point of $E \setminus N$ is separated by $\mathcal{S}''$ from y, we have that $K \setminus N \subset E^c$, and we see that the second alternative of (1) holds with $S' = S$ and $K' = K$, and conclude that $S \in \mathcal{S}$. □

Lemma 13 *Every sphere S from the family $\mathcal{S}$ from the previous Lemma is contained in ∂E.*

Proof. If S contains an interior point of E and $\varepsilon > 0$ is small enough, then any sphere S' such that $d(S,S') < \varepsilon$ also meets the interior of E and so by Lemma 11 any K-set K' about such S' meets the interior of E in a set that is not cap null. So none of $K' \cap E$ and $S \cap E$ is cap null (the latter again by Lemma 11), and so none of the alternatives from the statement (1) of Lemma 12 can hold. Similarly, or by exchanging the role of E and E^c we show that S cannot contain interior points of E^c. □

Theorem 14 *If C_1 and C_2 are two proper caps in X such that ∂C_1 and ∂C_2 are crossing then the set $E = C_1 \cap C_2$ does not belong to the Dynkin system generated by the caps.*

Proof. Let N be a cap null set and $\mathcal{S}$ a family of spheres with the properties from Lemma 12. From Lemma 13 we see that $S \subset \partial E$ for any $S \in \mathcal{S}$; since there is no such sphere, $\mathcal{S} = \emptyset$. But then the separation property (2) of Lemma 12 could hold only if one of the sets E, E^c were cap null, which is not the case because of Lemma 11. □

Remark 15 From Lemma 12 (together with its proof) and Lemma 13 we get the following strong necessary condition for a set $E \subset X$ from the Dynkin system generated by the caps:

There is a cap null set N and a closed (in the metric d) family $\mathcal{S}$ of pairwise non-crossing spheres such that each sphere $S \in \mathcal{S}$ is contained in ∂E and either in $E \cup N$ or in $E^c \cup N$, and every pair of points $x \in E \setminus (N \cup \bigcup_{S \in \mathcal{S}} S)$ and $y \in E^c \setminus (N \cup \bigcup_{S \in \mathcal{S}} S)$ is separated by some $S \in \mathcal{S}$.

Perhaps a similar condition with smaller negligible set N can be also sufficient.

References

[1] Davies, R. O., *Measures not approximable or not specifiable by means of balls*, Mathematika **18** (1971), 157-160.

[2] Jackson, S.; Mauldin, R. D., *On the σ-class generated by open balls*, submitted.

[3] Keleti, T., *The Dynkin system generated by the large balls of* $\mathbf{R}^n$, submitted.

[4] Mejlbro, L.; Preiss, D.; Tišer, J., *Positivity Principles in Geometrical Measure Theory*, Proc. of the Measure Theory Conference, Oberwolfach, 1990, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II – numero 28 – anno 1992, 163–167.

[5] Mejlbro, L.; Preiss, D.; Tišer, J., *Determination and differentiation of measures*, forthcoming book.

[6] Olejček, V., *Generation of a q-σ-algebra in the plane*, Proc. Conf. Topology and Measure V, Greifswald, 1988, 121-125.

[7] Olejček, V., *The σ-class generated by balls contains all Borel sets*, Proc. Amer. Math. Soc. **123** (1995), 3665-3675.

[8] Preiss, D., *Differentiation of measures in infinitely dimensional spaces*, Proc. Conf. Topology and Measure III, Greifswald, 1982, 201–207.

[9] Preiss, D.; Tišer, J., *Measures in Banach spaces are determined by their values on balls*, Mathematika **38** (1991), 391-397.

[10] Riss, E., *A scoopable Banach space that is isomorphic to ℓ_2*, Algebra i Analiz **7** (1995), no. 5, 204–224 (in Russian); English translation: St. Petersburg Math. J. **7** (1996), no. 5, 831–845.

[11] Riss, E., *On the scooping property of measures by means of disjoint balls*, Acta Univ. Carol., Math. Phys. **37**, No.2, 71-82 (1996).

[12] Zelený, M., *The Dynkin system generated by balls in $\mathbf{R}^d$ contains all Borel sets*, submitted.

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