Difference functions of periodic measurable functions *

Tamás Keleti †

December 20, 2004

Abstract

We investigate some problems of the following type: For which sets H is it true that if f is a function from the given class of periodic functions $\mathcal{F}$ and the difference functions $\Delta_h f(x) = f(x+h) - f(x)$ are in a given smaller class of functions $\mathcal{G}$ for every $h \in H$ then f itself must be in $\mathcal{G}$? Denoting the class of counter-example sets by $\mathfrak{H}(\mathcal{F}, \mathcal{G})$, that is $\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \left\{ H \subset \mathbb{R}/\mathbb{Z} : (\exists f \in \mathcal{F} \setminus \mathcal{G})\ (\forall h \in H)\ \Delta_h f \in \mathcal{G} \right\}$, we try to characterize $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ for some interesting classes of functions $\mathcal{F} \supset \mathcal{G}$. We study classes of measurable functions on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ that are invariant for changes on zero-sets (e.g. measurable functions, L_p, L_∞, essentially continuous functions, functions with absolute convergent Fourier series (ACF^*), essentially Lipschitz functions) and classes of continuous functions on $\mathbb{T}$ (e.g. continuous functions, continuous functions with absolute convergent Fourier series, Lipschitz functions). The classes $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ are often related to some classes of thin sets in harmonic analysis (e.g. $\mathfrak{H}(L_1, ACF^*)$ is the class of N-sets). Some results concerning the difference property and the weak difference property of these classes of functions are also obtained.

1991 *Mathematical Subject Classification.* Primary 28A20; Secondary 04A15, 26A99, 28A05, 39A70, 42A28, 43A46.

*The author wishes to express his gratitude to Professor Miklós Laczkovich for his advice and encouragement to the completion of a PhD dissertation [9] which this paper is based on.

†Supported by OTKA grant F 019468.

1 Introduction

In this paper we investigate problems of the following type:

Let f be a "nice" function. For which sets H is it true that

(∗) *if the difference functions $\Delta_h f(x) = f(x+h) - f(x)$ are "even nicer" for every $h \in H$ then f itself must be "even nicer"?*

1.1 Notation

We introduce the following notation. Let $\mathbb{G}$ be either the additive group of reals $\mathbb{R}$ or the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of functions on $\mathbb{G}$ with $\mathcal{F} \supset \mathcal{G}$. We denote by $\mathfrak{H}^0(\mathcal{F}, \mathcal{G})$ the class of those subsets H of $\mathbb{G}$, for which there exists $f \in \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}$ if and only if $h \in H$. That is,

$$\mathfrak{H}^0(\mathcal{F}, \mathcal{G}) = \Big\{ \{h \in \mathbb{G} : \Delta_h f \in \mathcal{G}\} : f \in \mathcal{F} \setminus \mathcal{G} \Big\}.$$

We denote by $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ the class of sets that can be covered by a set in $\mathfrak{H}^0(\mathcal{F}, \mathcal{G})$. Note that $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ is the class of sets $H \subset \mathbb{G}$ for which there exists $f \in \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}$ for any $h \in H$. That is,

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \Big\{ H \subset \mathbb{G} : (\exists f \in \mathcal{F} \setminus \mathcal{G})\ (\forall h \in H)\ \Delta_h f \in \mathcal{G} \Big\}.$$

Thus the family of sets satisfying (∗) is precisely the complement of $\mathfrak{H}(\mathcal{F}, \mathcal{G})$. Our goal is to characterize $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ for certain natural families of functions.

(The families of sets are always denoted by Gothic letters, the classes of functions are denoted by calligraphic letters or by capitals.)

In this paper we focus on the following families of (periodic) functions on $\mathbb{T}$: measurable functions (L_0), L_p functions, essentially bounded measurable functions (L_∞), continuous functions ($\mathcal{C}$), continuous functions with absolute convergent Fourier series (ACF) and Lipschitz functions (with exponent 1) (Lip^1). (Note that by $L_0, L_p, L_\infty, \mathcal{C}, ACF$ and Lip^1 we denote classes of functions on the circle group $\mathbb{T}$.)

The classes $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ are often related to some classes of thin sets in harmonic analysis. Now we define those classes that will arise in our results. Detailed explanation of this topic can be read in the monographs [2], [17], in the recent research papers [6] and [7] or in the recent topical survey [5].

A set $H \subset \mathbb{T}$ is called a *pseudo-Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists an $n_0(x)$ such that $|\sin q_n \pi x| < \varepsilon_n$ if $n \geq n_0(x)$.

A set $H \subset \mathbb{T}$ is called an *N-set* if there exists a trigonometric series that is absolutely convergent on H but is not absolutely convergent everywhere; that is, if there exist sequences (a_n) and (b_n) such that $\sum_{n=1}^{\infty}(|a_n|+|b_n|) = \infty$ but for any $x \in H$,

$$\sum_{n=1}^{\infty} \left(\, |a_n \cos(2\pi n x)| + |b_n \sin(2\pi n x)| \, \right) < \infty.$$

The family of pseudo-Dirichlet sets and N-sets are denoted by $\mathfrak{pD}$ and $\mathfrak{N}$, respectively.

We denote by $\mathfrak{F}_\sigma$ the family of those subsets of $\mathbb{T}$ that can be covered by a proper F_σ subgroup of $\mathbb{T}$.

It is known that

$$\mathfrak{pD} \subsetneq \mathfrak{N} \subsetneq \mathfrak{F}_\sigma.$$

(The inclusions are easy. For the (not too difficult example) for $\mathfrak{pD} \neq \mathfrak{N}$ see e.g. [7]. It is much more difficult to construct a set from $\mathfrak{F}_\sigma \setminus \mathfrak{N}$. M. Laczkovich and I. Ruzsa recently constructed such a set in [15].)

1.2 Known results. The difference property

In 1951 N. G. de Bruijn introduced the following notion in [3]: a class of real functions $\mathcal{F}$ is said to have the *difference property*, if any real function f such that, for each h, $\Delta_h f \in \mathcal{F}$, is of the form $f = g + G$, where $g \in \mathcal{F}$ and G is additive, that is $G(x+y) = G(x)+G(y)$ for any x and y. He proved that the class of continuous functions and the class of periodic continuous functions have the difference property. He also proved in [3] and [4] the difference property for the class of differentiable, analytic, absolutely continuous and Riemann-integrable functions. M. Laczkovich proved in [13] that the class of point-wise discontinuous functions and some related classes also have the difference property.

Since a measurable additive function is necessarily linear we have $\mathbb{G} \notin \mathfrak{H}(\mathcal{F}, \mathcal{G})$ if $\mathcal{F}$ is a class of measurable functions on $\mathbb{G} = \mathbb{R}$ or $\mathbb{T}$ and $\mathcal{G} \subset \mathcal{F}$ is a class of functions on $\mathbb{G}$ having the difference property and invariant under the addition with linear functions (e.g. $\mathcal{G}$ is any of the above mentioned classes).

As the next lemma shows, for periodic continuous functions the other implication is also true, which means that the notion of $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ is a kind of generalization of the difference property.

Lemma 1.1 *If $\mathcal{G} \subset \mathcal{C}$ and $\mathcal{G}$ is invariant under the addition with constants then the following statements are equivalent:*

(i) $\mathbb{T} \notin \mathfrak{H}(\mathcal{C}, \mathcal{G})$,

(ii) *$\mathcal{G}$ has the difference property.*

Proof.

(i)⇒(ii): Suppose that $\Delta_h f \in \mathcal{G}$ for any h. Then, since $\mathcal{G} \subset \mathcal{C}$ and $\mathcal{C}$ has the difference property, f can be written in the form $g + G$, where $g \in \mathcal{C}$ and G is additive. Thus, for any h, $\Delta_h f = \Delta_h g + C$, where C is a constant. Hence $\Delta_h g = \Delta_h f - C \in \mathcal{G}$ for any h, which implies - using $\mathbb{T} \notin \mathfrak{H}(\mathcal{C}, \mathcal{G})$ - that $g \in \mathcal{G}$.

(ii)⇒(i): It is obvious by the previous observation. □

All of the results above concerned the case of $H = \mathbb{G}$. As far as I know, the first result answering a more general problem is the following:

Theorem 1.2 (Balcerzak, Buczolich and Laczkovich, 1997, [1]) *For any subset $H \subset \mathbb{T}$, the following statements are equivalent:*

(i) *If $f : \mathbb{T} \to \mathbb{R}$ is continuous and $\Delta_h f \in \mathrm{Lip}^1$ for every $h \in H$ then $f \in \mathrm{Lip}^1$.*

(ii) *There is no proper F_σ subgroup of $\mathbb{T}$ containing H.*

That is, with our notation

$$\mathfrak{H}(\mathcal{C}, \mathrm{Lip}^1) = \mathfrak{F}_\sigma.$$

Another result of this type:

Theorem 1.3 (1997, [8]) *For any pseudo-Dirichlet set H there exists a periodic function $f \in L_2 \setminus L_\infty$ for which $\Delta_h f$ is continuous for any $h \in H$.*

Thus, for any class $\mathcal{C} \subset \mathcal{G} \subset L_\infty$, we have

$$\mathfrak{H}(L_2, \mathcal{G}) \supset \mathfrak{p}\mathfrak{D}.$$

We will generalize these results in Sections 3 and 5.

1.3 Preliminary results

The following easy facts will be used frequently.

Lemma 1.4 *If $\mathcal{F} \supset \mathcal{G}$ and $\mathcal{G}$ is a translation invariant group of functions on $\mathbb{T}$ (with pointwise addition), then each element of $\mathfrak{H}^0(\mathcal{F},\mathcal{G})$ is a subgroup of $\mathbb{T}$.*

(We say that $\mathcal{G}$ is translation invariant, *if for any $g(x) \in \mathcal{G}$ and $a \in \mathbb{T}$, we have $g(x+a) \in \mathcal{G}$.)*

Proof. By definition

$$\Delta_{-h} f(x) = f(x-h) - f(x) = -\Delta_h f(x-h),$$

thus if $\Delta_h f \in \mathcal{G}$ then also $\Delta_{-h} f \in \mathcal{G}$. In addition

$$\Delta_{h_1+h_2} f(x) = \Delta_{h_2} f(x+h_1) + \Delta_{h_1} f(x),$$

therefore if $\Delta_{h_1} f, \Delta_{h_2} f \in \mathcal{G}$ then also $\Delta_{h_1+h_2} f \in \mathcal{G}$. □

Lemma 1.5 (Monotonicity Lemma) *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{G}$ then*

$$\mathfrak{H}^0(\mathcal{F}_1,\mathcal{G}) \supset \mathfrak{H}^0(\mathcal{F}_2,\mathcal{G}) \quad \textit{and} \quad \mathfrak{H}(\mathcal{F}_1,\mathcal{G}) \supset \mathfrak{H}(\mathcal{F}_2,\mathcal{G}). \ \square$$

Lemma 1.6 (Triangle inequality) *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ then*

$$\mathfrak{H}(\mathcal{F}_1,\mathcal{F}_3) \subset \mathfrak{H}(\mathcal{F}_1,\mathcal{F}_2) \ \cup \ \mathfrak{H}(\mathcal{F}_2,\mathcal{F}_3).$$

Proof. Suppose that $H \in \mathfrak{H}(\mathcal{F}_1,\mathcal{F}_3)$ but $H \notin \mathfrak{H}(\mathcal{F}_1,\mathcal{F}_2)$ and $H \notin \mathfrak{H}(\mathcal{F}_2,\mathcal{F}_3)$. Then $H \in \mathfrak{H}(\mathcal{F}_1,\mathcal{F}_3)$ implies that there exists a function $f \in \mathcal{F}_1 \setminus \mathcal{F}_3$ such that $\Delta_h f \in \mathcal{F}_3$ for any $h \in H$. Since $H \notin \mathfrak{H}(\mathcal{F}_1,\mathcal{F}_2)$ and $\mathcal{F}_3 \subset \mathcal{F}_2$, f cannot be in $f \in \mathcal{F}_1 \setminus \mathcal{F}_2$, therefore $f \in \mathcal{F}_2 \setminus \mathcal{F}_3$, which contradicts $H \notin \mathfrak{H}(\mathcal{F}_2,\mathcal{F}_3)$. □

Lemma 1.7 *If $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3$ and $\mathfrak{H}(\mathcal{F}_1,\mathcal{F}_2) \subset \mathfrak{H}(\mathcal{F}_2,\mathcal{F}_3)$ then*

$$\mathfrak{H}(\mathcal{F}_1,\mathcal{F}_3) = \mathfrak{H}(\mathcal{F}_2,\mathcal{F}_3).$$

Proof. This is trivial from Lemma 1.5 and Lemma 1.6. □

2 Changes on null-sets. Essentially continuous functions

Notation 2.1 If $\mathcal{F}$ is a class of functions we denote by $\mathcal{F}^*$ the class of those functions that are equal to a function in $\mathcal{F}$ almost everywhere.

If the elements of $\mathcal{F}$ are called P functions, where P is an arbitrary property (e.g. P=continuous) then we will call the functions in $\mathcal{F}^*$ *essentially* P functions.

In this section we investigate what happens if we replace a class of functions $\mathcal{F}$ by $\mathcal{F}^*$. We will see that in the most important cases the corresponding class $\mathfrak{H}$ either remains the same or becomes much more interesting.

The following lemma is obvious.

Lemma 2.2 *If $\mathcal{G} \subset \mathcal{C}$ then $\mathcal{G}^* \cap \mathcal{C} = \mathcal{G}$.* $\square$

Proposition 2.3 *If $\mathcal{C} \supset \mathcal{F} \supset \mathcal{G}$ then*

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \mathfrak{H}(\mathcal{F}, \mathcal{G}^*) = \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*).$$

Proof. $\underline{\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{H}(\mathcal{F}, \mathcal{G}^*)}$: If $H \in \mathfrak{H}(\mathcal{F}, \mathcal{G})$ then there exists a function $f \in \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}$ for any $h \in H$. Applying Lemma 2.2, we get that $f \notin \mathcal{G}^*$. Therefore $f \in \mathcal{F} \setminus \mathcal{G}^*$ and $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$, which shows that $H \in \mathfrak{H}(\mathcal{F}, \mathcal{G}^*)$.

$\underline{\mathfrak{H}(\mathcal{F}, \mathcal{G}) \supset \mathfrak{H}(\mathcal{F}, \mathcal{G}^*)}$: If $H \in \mathfrak{H}(\mathcal{F}, \mathcal{G}^*)$ then there exists a function $f \in \mathcal{F} \setminus \mathcal{G}^* \subset \mathcal{F} \setminus \mathcal{G}$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$. Since $f \in \mathcal{F} \subset \mathcal{C}$ we get $\Delta_h f \in \mathcal{C}$. Applying Lemma 2.2, we get that $\Delta_h f \in \mathcal{G}$. Therefore $f \in \mathcal{F} \setminus \mathcal{G}$ and $\Delta_h f \in \mathcal{G}$ for any $h \in H$, which shows that $H \in \mathfrak{H}(\mathcal{F}, \mathcal{G})$.

$\underline{\mathfrak{H}(\mathcal{F}, \mathcal{G}^*) \subset \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)}$: This follows from the monotonicity-lemma.

$\underline{\mathfrak{H}(\mathcal{F}, \mathcal{G}^*) \supset \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)}$: If $H \in \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$ then there exists a function $f \in \mathcal{F}^* \setminus \mathcal{G}^*$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in H$. Since $f \in \mathcal{F}^*$ there exists a function $\tilde{f} \in \mathcal{F}$ such that $f = \tilde{f}$ a.e. Since $f \notin \mathcal{G}^*$ we get $\tilde{f} \notin \mathcal{G}^*$, hence $\tilde{f} \in \mathcal{F} \setminus \mathcal{G}^*$. On the other hand $\Delta_h f \in \mathcal{G}^*$ implies that $\Delta_h \tilde{f} \in \mathcal{G}^*$. Therefore $H \in \mathfrak{H}(\mathcal{F}, \mathcal{G}^*)$. $\square$

Proposition 2.4 *If $\mathcal{G} \subset \mathcal{F} \subset L_0$, $\mathcal{G} \subset \mathcal{C}$ and $\mathcal{G}$ contains the constant 0 function, then*

$$\mathfrak{H}^0(\mathcal{F}^*, \mathcal{G}) \supset \{\text{additive subgroups of zero measure}\}.$$

Proof. Let A be an additive subgroup with zero measure. Let f be its characteristic function.

Since $f = 0$ a.e. and $0 \in \mathcal{G} \subset \mathcal{F}$ we get $f \in \mathcal{F}^*$. If $a \in A$ then $\Delta_a f = 0 \in \mathcal{G}$. If $a \notin A$ then $\Delta_a f$ is a non-constant function with finite range, so it cannot be continuous, hence it is not in $\mathcal{G}$. Therefore f shows that $A \in \mathfrak{H}^0(\mathcal{F}^*, \mathcal{G})$. □

Remark 2.5 It is also proved in the author's PhD thesis ([9]) that if $\mathcal{G}$ is a closed, translation invariant subspace of $\mathcal{C}$ then equality holds in Proposition 2.4.

In the sequel we will work with classes of functions of the following two types:

(i) classes of measurable functions that are invariant for changes on zero-sets (that is, $\mathcal{F} = \mathcal{F}^*$);

(ii) classes of continuous functions that contain the constant 0 function.

Instead of $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ we will usually investigate $\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$. If $\mathcal{F}$ and $\mathcal{G}$ are both of type (i), then these classes of sets are trivially the same, if $\mathcal{F}$ and $\mathcal{G}$ are both of type (ii) then Proposition 2.3 shows that these classes of sets are equal.

If $\mathcal{F}$ is of type (i) and $\mathcal{G}$ is of type (ii) then these classes are usually not equal (we will show that $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ contains $\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$), but in this case, as Proposition 2.4 shows, $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ is "too big". In this case it is much more interesting to investigate $\mathfrak{H}(\mathcal{F}, \mathcal{G}^*)$, which is the same as $\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$.

The following lemma was proved in [8] (Lemma 2). (In fact, the last statement is not stated in [8], however, that easily follows from the proof.)

Lemma 2.6 *Let A be an additive subgroup of $\mathbb{R}$ and let S be a dense union of translated copies of A. Suppose that we have a function $h : \mathbb{R} \to \mathbb{R}$ and continuous functions $l_a : \mathbb{R} \to \mathbb{R}$ for all $a \in A$ such that $\Delta_a h|S = l_a|S$ for any $a \in A$.*

Then there exists a function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{h}|S = h|S$ and $\Delta_a \tilde{h} = l_a$ for every $a \in A$.

Moreover, if h is bounded, then we can choose $\tilde{h}$ to be also bounded. □

Main Lemma 2.7 *Suppose that $H \subset \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is a measurable function and $\Delta_h f$ is essentially continuous for any $h \in H$.*

Then there exists a function $\tilde{f}$ such that $\tilde{f} = f$ a.e. and $\Delta_h f$ is continuous for any $h \in H$.

Moreover, if f is bounded, then we can choose $\tilde{f}$ to be also bounded.

Proof. Let A be the additive subgroup of $\mathbb{R}$ generated by H. Then clearly $\Delta_a f$ is essentially continuous also for any $a \in A$. Thus for each $a \in A$ there exists a continuous function l_a such that $\Delta_a f = l_a$ a.e. Then

$$f(x+a) = f(x) + l_a(x) \text{ a.e. (for any fixed } a \in A). \tag{1}$$

Let

$$S = \{x : f \text{ has a finite approximative limit at } x\}.$$

Since f is a measurable function the set S has full measure.

For any $x \in S$, the right-hand-side of (1) has a finite approximative limit at x, so the left-hand-side, which is a.e. equal to it, also has a finite approximative limit at x. That is, if $x \in S$ and $a \in A$ then $x + a \in S$. Therefore S is a dense (since it has full measure) union of translated copies of A.

Let

$$f_1(x) = \begin{cases} \text{limappr}_x f & \text{if } x \in S \\ f(x) & \text{if } x \notin S. \end{cases}$$

If f is bounded then so is f_1. Since f is measurable it is almost everywhere approximately continuous, so $f_1 = f$ a.e. This implies that their approximative limits are equal everywhere. Thus for any $x \in S$ we get $f_1(x) = \text{limappr}_x f = \text{limappr}_x f_1$, which implies that f_1 (and thus also $\Delta_a f_1$) is approximately continuous at the points of S. On the other hand $\Delta_a f_1 = \Delta_a f$ a.e. and $\Delta_a f = l_a$ a.e, so $\Delta_a f_1 = l_a$ a.e.

Hence for any $x \in S$ and $a \in A$ we get

$$\Delta_a f_1(x) = \text{limappr}_x \Delta_a f_1(x) = \text{limappr}_x l_a = l_a(x).$$

Now applying the previous lemma, changing f_1 on the complement of S, we can get a function $\tilde{f}$ such that $\Delta_a \tilde{f} = l_a$ for any $a \in A$. Thus $\tilde{f} = f$ a.e. (since $\tilde{f} = f_1$ on S, S has full measure and $f_1 = f$ a.e.) and $\Delta_h \tilde{f}$ is continuous for any $h \in H \subset A$. Moreover $\tilde{f}$ is bounded, if f is bounded. □

Corollary 2.8 *If $\mathcal{G} \subset \mathcal{F} \subset L_0$ and $\mathcal{G} \subset \mathcal{C}$, then*

$$\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*) \subset \mathfrak{H}(\mathcal{F}^*, \mathcal{G}). \quad \square$$

Theorem 2.9 *If $f : \mathbb{R} \to \mathbb{R}$ is measurable, $\Delta_h f$ is essentially continuous for any $h \in \mathbb{R}$ then f is also essentially continuous.*

Proof. According to the Main Lemma, there exists a function $\tilde{f}$ such that $\tilde{f} = f$ a.e. and $\Delta_h \tilde{f}$ is continuous for any $h \in \mathbb{R}$. Then using the difference property of the continuous functions (see Subsection 1.2) we get that $\tilde{f}$ is a sum of a continuous and an additive functions. But since $\tilde{f}$ is measurable, this implies that $\tilde{f}$ is continuous, so f is essentially continuous. $\square$

Remark 2.10 For periodic functions this theorem is the first step for a stronger result. We will prove (Theorem 4.7) that if f is a measurable function on $\mathbb{T}$ and $\Delta_h f$ is essentially continuous for any $h \in H$, and H cannot be covered by a proper F_σ-subgroup of $\mathbb{T}$, then f is essentially continuous.

At this point one can hope that the class of essentially continuous functions has the difference property; that is, for any $f : \mathbb{R} \to \mathbb{R}$, if $\Delta_h f \in \mathcal{C}^*$ for any $h \in \mathbb{R}$ then f is a sum of an essentially continuous and an additive functions. However, this is not the case. More precisely the following is true:

Theorem 2.11 *Supposing the continuum hypothesis, the class of essentially continuous functions does not have the difference property.*

Proof. Assuming CH, Sierpiński ([16]) constructed a non-measurable function $S : \mathbb{R} \to \{0, 1\}$ such that for any fixed $h \in \mathbb{R}$, $\Delta_h S(x) = 0$ with exception of at most a countable number of x-values.

Then clearly $\Delta_h S \in \mathcal{C}^*$ for any $h \in \mathbb{R}$. But if S was the sum of an essentially continuous and an additive functions, then the additive function would be essentially bounded on any interval, which would mean that it is linear. Then S would be essentially continuous but S is not measurable. $\square$

However, the class $\mathcal{C}^*$ has a weaker property. We say that a class $\mathcal{F}$ has the *weak difference property* if every function $f : \mathbb{G} \to \mathbb{R}$ for which $\Delta_h f \in \mathcal{F}$ for every $h \in \mathbb{G}$ admits a decomposition $f = g + H + S$ with $g \in \mathcal{F}$, H additive, and S satisfying the condition that for every $h \in \mathbb{G}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbb{G}$.

Lemma 2.12 *Suppose that (i) $\mathcal{F} \supset \mathcal{G}$ are classes of measurable functions on $\mathbb{G}$ (where $\mathbb{G} = \mathbb{T}$ or $\mathbb{R}$), (ii) $\mathcal{G}$ is a group that contains the constant functions and the linear functions, and (iii) $\mathcal{F}^*$ has the weak difference property.*

Then $\mathcal{G}^*$ has the weak difference property if and only if $\mathbb{G} \notin \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$.

Proof. Assume that $\mathcal{G}^*$ has the weak difference property but $\mathbb{G} \in \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$. Then there exists a function $f \in \mathcal{F}^* \setminus \mathcal{G}^*$ such that $\Delta_h f \in \mathcal{G}^*$ for any $h \in \mathbb{G}$. Since $\mathcal{G}^*$ has the weak difference property, this implies that $f = g + H + S$ where $g \in \mathcal{G}^*$, H is additive, and S satisfies the condition that for every $h \in \mathbb{G}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbb{G}$.

Let $l = f - g = H + S$. Then l is measurable and $\Delta_h l$ is constant a.e. for any $h \in \mathbb{G}$. Thus, by the Main Lemma (2.7), there exists a function $\tilde{l}$ such that $\tilde{l} = l$ a.e. and $\Delta_h \tilde{l}$ is constant everywhere. Then $\tilde{l} - \tilde{l}(0)$ is a measurable additive function, so $\tilde{l}$ is linear, thus $\tilde{l} \in \mathcal{G}$. Since $f = g + \tilde{l}$ a.e., $g \in \mathcal{G}^*$ and $\mathcal{G}$ is a group, this implies that $f \in \mathcal{G}^*$, which is a contradiction.

Now we prove that if $\mathbb{G} \notin \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$ then $\mathcal{G}^*$ has the weak difference property. Suppose that for a function $f : \mathbb{G} \to \mathbb{R}$, $\Delta_h f \in \mathcal{G}^*$ for every $h \in \mathbb{G}$. Then, since $\mathcal{G}^* \subset \mathcal{F}^*$, f has a decomposition $f = g + H + S$ with $g \in \mathcal{F}^*$, H additive, and S satisfying the condition that for every $h \in \mathbb{G}$, $\Delta_h S(x) = 0$ holds for a.e. $x \in \mathbb{G}$. Then $\Delta_h f = \Delta_h g + \Delta_h H + \Delta_h S$. Since $\Delta_h f \in \mathcal{G}^*$, $\Delta_h S = 0$ a.e. and $\Delta_h H$ is constant, this implies that also $\Delta_h g \in \mathcal{G}^*$ for any $h \in \mathbb{G}$. Therefore, since $\mathbb{G} \notin \mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$, $g \in \mathcal{G}^*$. $\square$

Theorem 2.13 *The class of essentially continuous functions has the weak difference property.*

Proof. In [12] M. Laczkovich proved that the class of measurable functions has the weak difference property. Then, by Lemma 2.12, Theorem 2.9 implies that $\mathcal{C}^*$ also has the weak difference property. $\square$

Notation 2.14 For an $f : \mathbb{G} \to \mathbb{R}$ function, where $\mathbb{G} = \mathbb{R}$ or $\mathbb{T}$, we denote by H_f the set of h's for which $\Delta_h f$ is continuous.

Proposition 2.15 *If a function $g : \mathbb{R} \to \mathbb{R}$ has a point of continuity and H_g is a dense set, then g must be continuous everywhere.*

Proof. Let $\omega(x)$ be the oscillation of g at x. Since the function $\omega(x)$ is upper semi-continuous, the sets of the form $\{x : \omega(x) \geq c\}$ are closed for any $c \in \mathbb{R}$. On the other hand $\omega(x)$ is periodic modulo h for any $h \in H_g$, since $g(x+h) = \Delta_h g(x) + g(x)$, and $\Delta_h g$ is continuous everywhere.

Therefore for any $c \in \mathbb{R}$ the set $\{x : \omega(x) \geq c\}$ is closed and is periodic modulo a dense set, so these sets must be either empty or the whole real line, which implies that $\omega(x)$ is constant. Since g has a point of continuity, this constant must be 0, which means that g is continuous. □

Proposition 2.16 *If $f : \mathbb{R} \to \mathbb{R}$ is measurable but not essentially continuous and H_f is dense, then*

$$\limsup_x f = +\infty \quad \text{and} \quad \liminf_x f = -\infty \qquad (\forall x \in \mathbb{R}).$$

Proof. We prove the $\limsup f = +\infty$ part of the statement, the proof of the other statement is the same.

We shall use the notation $\overline{f}(x) = \max\{f(x), \limsup_x f\}$. Since $f(x+h) = \Delta_h f(x) + f(x)$ and $\Delta_h f$ is continuous for $h \in H_f$ it follows that $\overline{f} - f$ is periodic modulo each $h \in H_f$. Thus if $\overline{f}(x_0) = +\infty$ for any $x_0 \in \mathbb{R}$ then $\limsup_x f = +\infty$ on a dense set, which implies that $\limsup_x f = +\infty$ everywhere. Therefore we can assume that $\overline{f}$ is finite everywhere.

For a fixed $h \in H_f$, the function $\overline{f} - f$ is periodic modulo h, so $\overline{f}(x+h) - f(x+h) = \overline{f}(x) - f(x)$, which implies that $\Delta_h \overline{f} = \Delta_h f$. Therefore for any $h \in H_f$, $\Delta_h \overline{f}$ is also continuous. Thus $H_{\overline{f}}$ is also dense. On the other hand $\overline{f}$ is upper semi-continuous, so it is Baire-1, so it has a point of continuity. Then according to Proposition 2.15, $\overline{f}$ is continuous.

Since $\overline{f} - f$ is measurable and its periods form a dense set, $\overline{f} - f$ is constant a.e. Thus, since $\overline{f}$ is continuous, f is essentially continuous, contradicting our assumption. □

Theorem 2.17 *If $f : \mathbb{R} \to \mathbb{R}$ is measurable and essentially bounded and $\Delta_h f$ is essentially continuous for a dense set of h-s, then f is essentially continuous.*

Proof. Let

$$H = \{h : \Delta_h f \text{ is essentially continuous}\}.$$

Since f is essentially bounded there exists an f_1 such that $f_1 = f$ a.e. and f_1 is bounded. Then for any $h \in H$, $\Delta_h f_1$ is also essentially continuous. Applying the Main Lemma we can take a bounded function $\tilde{f}$ such that

$\tilde{f} = f_1$ a.e. and $\Delta_h \tilde{f}$ is continuous for any $h \in H$. Since H is dense, the last proposition shows that this can happen only if $\tilde{f}$ is essentially continuous. But then f, which is almost everywhere equal to $\tilde{f}$, is also essentially continuous. □

Corollary 2.18

$$\mathfrak{H}^0(L_\infty, \mathcal{C}^*) = \{\text{finite subgroups of } \mathbb{T}\},$$

$$\mathfrak{H}(L_\infty, \mathcal{C}^*) = \{\text{finite subsets of } \mathbb{T} \cap \mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of rational numbers.

Proof. Since the subsets of $\mathbb{T}$ that can be covered by a finite subgroup of $\mathbb{T}$ are the finite subsets of $\mathbb{T} \cap \mathbb{Q}$ it is enough to prove the first equality.

- $\subset$: This is an immediate consequence of the previous theorem, Lemma 1.4 and the fact that an infinite subgroup of $\mathbb{T}$ is dense.
- $\supset$: Let G be a finite subgroup of $\mathbb{T}$. Then it is easy to see that G is of the form $G = \{0, 1/n, 2/n, \ldots, (n-1)/n\}$. Let $f(x) = \operatorname{sgn}(\sin(2\pi n x))$. Then clearly $f \in L_\infty \setminus \mathcal{C}^*$ and $\{h : \Delta_h f \in \mathcal{C}^*\} = G$. □

3 Not essentially bounded periodic measurable functions with many continuous difference functions.

$$\mathfrak{H}(L_p, ACF^*) = \mathfrak{N}$$

In this section we generalize the main results of [8] and we prove that for any $p \geq 1$, $\mathfrak{H}(L_p, ACF^*) = \mathfrak{N}$.

Lemma 3.1 *If $d_1 \geq d_2 \geq \ldots \geq 0$ and $\sum d_n = \infty$, then $\sum \min(d_n, 1/n) = \infty$.*

Proof. We can assume that $d_n > 1/n$ for infinitely many n, since otherwise $\min(d_n, 1/n) = d_n$ for n large enough.

Then we can choose a subsequence d_{n_k} such that $n_k \geq 2n_{k-1}$ and $d_{n_k} > 1/n_k$ for every k. Then

$$\begin{aligned}\sum \min(d_n, 1/n) &= \sum_k \sum_{m=n_{k-1}+1}^{n_k} \min(d_m, 1/m) \geq \sum_k \sum_{m=n_{k-1}+1}^{n_k} 1/n_k = \\ &= \sum_k \frac{n_k - n_{k-1}}{n_k} \geq \sum_k \frac{1}{2} = \infty. \quad \square\end{aligned}$$

Lemma 3.2 *If $\sum a_n$ is a nonnegative divergent series then, by decreasing a_n for some indices n, we can get a nonnegative divergent series $\sum b_n$ for which $\sum b_n^q < \infty$ for every $q > 1$.*

Proof. If $a_n \to 0$ then we can rearrange (a_n) such that $a_{\phi(1)} \geq a_{\phi(2)} \geq \ldots$ where ϕ is a permutation of $\mathbb{N}$. In this case let $b_{\phi(n)} = \min(a_{\phi(n)}, 1/n)$. Then, applying the previous lemma for $d_n = a_{\phi(n)}$, we get $\sum b_n = \sum b_{\phi(n)} = \infty$. On the other hand $\sum b_n^q = \sum b_{\phi(n)}^q < \infty$ for every $q > 1$, since $b_{\phi(n)} \leq 1/n$. Furthermore clearly $0 \leq b_n \leq a_n$ $(n = 1, 2, \ldots)$.

If $a_n \not\to 0$ then there exists an $\varepsilon > 0$ and a subsequence a_{n_m} such that $a_{n_m} > \varepsilon$. Then let $b_{n_m} = \varepsilon/m$ and let the other terms of the sequence (b_n) be 0. Then in this case clearly also $0 \leq b_n \leq a_n$ $(n = 1, 2, \ldots)$, $\sum b_n = \infty$ and $\sum b_n^q < \infty$ for every $q > 1$. $\square$

Theorem 3.3 *For every N-set $H \subset \mathbb{R}$ there exists a function $f : \mathbb{R} \to \mathbb{R}$ periodic modulo 1 such that $f \in L_p$ for every finite p but $f \notin L_\infty$, and $\Delta_h f$ is continuous and has an absolutely convergent Fourier series for every $h \in H$.*

Proof. It is known (see e.g. [17] Vol. I, p. 236) that if H is an N-set, then it is also an N_s-set; that is, there exists a nonnegative divergent series $\sum b_n$ such that

$$\sum b_n |\sin \pi n h| < \infty \qquad (\forall h \in H). \tag{2}$$

Applying Lemma 3.2, we can also assume that $\sum b_n^q < \infty$ for every $q > 1$.

Let A denote the set of all h-s for which (2) holds. It is easy to see that A is an additive subgroup of $\mathbb{R}$ and $H \subset A$. Let $\tilde{f}$ be a modulo 1 periodic complex valued function with the Fourier series

$$\tilde{f}(x) \sim \sum_{n=1}^{\infty} b_n e^{2\pi inx}.$$

According to the Riesz-Fischer theorem, $\sum b_n^q < \infty$ for every $q > 1$ implies that such a function exists in L_2. Moreover, this condition implies that this function is in L_p for every $p > 0$ (see e.g. [17], proof of the Hausdorff-Young theorem, Vol. II, p. 101-103). Let $\overline{f} = \operatorname{Re} \tilde{f}$. Then clearly also $\overline{f} \in \cap_{p>0} L_p$.

It is known and easy to prove using Fejér means (see e.g. [2] IV. §2 Theorem 1, p. 277) that if a bounded real even function has Fourier series with nonnegative coefficients (c_n) then $\sum c_n < \infty$. Since $\overline{f}(x) \sim \sum_{n=1}^{\infty} b_n \cos(2\pi nx)$, and $\sum b_n = \infty$ we get that $\overline{f} \notin L_\infty$.

For a fixed $h \in A$ the Fourier series of $\tilde{f}(x+h)$ is

$$\tilde{f}(x+h) \sim \sum_{n=1}^{\infty} (b_n e^{2\pi inh}) e^{2\pi inx},$$

so

$$\Delta_h \tilde{f}(x) \sim \sum_{n=1}^{\infty} b_n (e^{2\pi inh} - 1) e^{2\pi inx}. \tag{3}$$

On the other hand

$$\left| b_n (e^{2\pi inh} - 1) e^{2\pi inx} \right| = 2 b_n |\sin \pi nh|.$$

Thus (2) implies that the right-hand side of (3) is uniformly convergent, so denoting it by $\tilde{l}_h(x)$, the function $\tilde{l}_h(x)$ is continuous on $\mathbb{R}$.

Let S be the set of points x where the averages of the partial sums of the Fourier series (the Fejér means) of $\tilde{f}$ converge to $\tilde{f}(x)$. According to Lebesgue's theorem S contains the Lebesgue points of $\tilde{f}$, so its complement is a null-set. Changing $\tilde{f}$ on this null-set we can make $\tilde{f}(x)$ to be equal to the limit of the Fejér means at each points where it exists, so we can assume that S is also the set of points where the Fejér means converge.

Since the Fejér means of $\tilde{l}_h$ converge to $\tilde{l}_h(x)$ everywhere, the Fejér means of $\tilde{f}(x)$ and $\tilde{f}(x+h)$ converge simultaneously, thus $x \in S$ if and only if

$x+h \in S$. Therefore S is a dense union of translated copies of A. If $x \in S$ then, according to (3), $\Delta_h \tilde{f}(x)$ and $\tilde{l}_h(x)$ are the limits of the averages of the partial sums of the same Fourier series, thus $\Delta_h \tilde{f}(x) = \tilde{l}_h(x)$ if $x \in S$. Therefore denoting the real part of $\tilde{l}_h$ by l_h we get

$$\Delta_h \overline{f}(x) = l_h(x) \qquad (x \in S,\ h \in A).$$

Now applying Lemma 2.6, there exists a function $f(x)$ on $\mathbb{R}$ such that $f|S = \overline{f}|S$ and

$$\Delta_h f(x) = l_h(x) \qquad (x \in \mathbb{R},\ h \in A).$$

In particular $\Delta_1 f(x) = l_1(x) = 0$, which implies that f is periodic modulo 1; for every $h \in H \subset A$ we get $\Delta_h f = l_h$, which shows that $\Delta_h f$ is continuous for every $h \in H$. Since f and $\overline{f}$ are equal almost everywhere and $\overline{f} \in (\cap_{p>0} L_p) \setminus L_\infty$ we get $f \in (\cap_{p>0} L_p) \setminus L_\infty$. □

The previous theorem is a generalization of a result of [8] (Theorem 1). The same easy way as in [8] Theorem 2 follows from Theorem 1 we get the following generalization of Theorem 2 of [8]:

Corollary 3.4 *For any N-set $H \subset \mathbb{R}$ there exists a modulo 1 periodic function $h \in (\cap_{p>0} L_p(\mathbb{R})) \setminus L_\infty(\mathbb{R})$ and there are modulo α periodic functions $g_\alpha \in (\cap_{p>0} L_p(\mathbb{R}))$ for all $\alpha \in H$ such that $g_\alpha + h$ is continuous for all $\alpha \in H$.*

(Here by $L_p(\mathbb{R})$ we mean the class of those measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which $|f|^p$ has a finite integral on any finite interval; $L_\infty(\mathbb{R})$ denotes the class of essentially bounded measurable $\mathbb{R} \to \mathbb{R}$ functions.)

Notation 3.5 We recall that we denote by ACF the class of continuous functions with absolute convergent Fourier series on $\mathbb{T}$.

We use the notation $\mathfrak{N}$ for the class of N-subsets of $\mathbb{T}$.

Corollary 3.6 *If $ACF \subset \mathcal{F} \subset L_\infty$ and $0 < p < \infty$ then*

$$\mathfrak{H}(L_p, \mathcal{F}^*) \supset \mathfrak{N}.$$

Proof. This is an immediate consequence of Theorem 3.3. □

Theorem 3.7

$$\mathfrak{H}(L_1, ACF^*) \subset \mathfrak{N}.$$

Proof. Let $H \in \mathfrak{H}(L_1, ACF^*)$. Then there exists a function $f \in L_1 \setminus ACF^*$ such that $\Delta_h f \in ACF^*$ for every $h \in H$. Let the Fourier series of f be

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \qquad (c_{-k} = \overline{c}_k).$$

Since the Fourier series of f is not absolutely convergent we get $\sum_{k=1}^{\infty} |c_k| = \infty$. It is easy to see that the Fourier series of $\Delta_h f$ is

$$\Delta_h f \sim \sum_{k=-\infty}^{\infty} c_k (e^{2\pi ikh} - 1) e^{2\pi ikx}.$$

Let

$$E = \{h \in \mathbb{T} \ : \ \sum_{k=1}^{\infty} |c_k||e^{2\pi ikh} - 1| < \infty\}.$$

Then, since $\Delta_h f$ has an absolutely convergent Fourier series for every $h \in H$, we get $H \subset E$.

In [6] B. Host, J.-F. Méla and F. Parreau call a set of type

$$\{h \in \mathbb{T} \ : \ \sum_{j=0}^{\infty} a_j |e^{2\pi in_j h} - 1| < \infty\} \tag{4}$$

an H_1 group if n_j is a sequence of positive integers and $a_j \geq 0$ (p. 44, 2.3.1). They proved that if $\sum_{j=0}^{\infty} a_j = \infty$, then the H_1 group defined by (4) is a proper subgroup of $\mathbb{T}$. They also proved that, for a Borel subset of $\mathbb{T}$, it is equivalent being an N-set and being contained in an H_1 proper subgroup.

Using these facts and notation, E is an H_1 proper subgroup of $\mathbb{T}$, thus (since E is clearly an F_σ set so it is also Borel) E is an N-set. Since $H \subset E$ we get that H is also an N-set. □

Corollary 3.8 *For every $p \geq 1$*

$$\mathfrak{H}(L_p, ACF^*) = \mathfrak{N}.$$

Proof. This is trivial from Corollary 3.6, Theorem 3.7 and from the Monotonicity Lemma. □

Corollary 3.9 *ACF has the difference property.*

Proof. By Proposition 2.3, the Monotonicity Lemma and Theorem 3.7,

$$\mathfrak{H}(\mathcal{C}, ACF) = \mathfrak{H}(\mathcal{C}^*, ACF^*) \subset \mathfrak{H}(L_1, ACF^*) \subset \mathfrak{N}.$$

Hence $\mathbb{T} \notin \mathfrak{H}(\mathcal{C}, ACF)$, so according to Lemma 1.1, ACF has the difference property. □

4 $\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{F}_\sigma$ for the classes $L_0, L_p, L_\infty, \mathcal{C}^*$, ACF^* and $(\mathrm{Lip}^\alpha)^*$

Consider the following classes of functions

$$L_0 \supset L_p \supset L_\infty \supset \mathcal{C}^* \supset ACF^* \text{ and } (\mathrm{Lip}^\alpha)^*.$$

(If $\alpha > \frac{1}{2}$ then, by a theorem of S. Bernstein (see e.g. [17] Vol. I, p. 240), we have also $ACF \supset \mathrm{Lip}^\alpha$.) In this section we prove that for any pair of these classes of functions we have

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{F}_\sigma \qquad (\mathcal{F} \supset \mathcal{G}).$$

(We recall that $\mathfrak{F}_\sigma$ is the class of subsets of $\mathbb{T}$ that can be covered by a proper F_σ subgroup of $\mathbb{T}$.) By the Monotonicity Lemma, it is enough to prove this for $\mathcal{F} = L_0$. If $\mathcal{G} \subset \mathcal{F} \subset \mathcal{C}$ then, by Proposition 2.3, everything remains the same without $*$; that is, we have the same results for $\mathcal{C}, ACF$ and Lip^α.

We will need the following well-known lemma:

Lemma 4.1 *If $f : \mathbb{T} \to \mathbb{R}$ is a measurable function and (a_n) is a sequence of reals converging to 0, then we can choose a subsequence (a_{n_k}) such that*

$$\lim_{k \to \infty} f(x + a_{n_k}) = f(x) \qquad \text{for a. e. } x \in \mathbb{T}.$$

Proposition 4.2 *The sets in $\mathfrak{H}^0(L_0, L_p)$, for any $0 < p \leq \infty$, are F_σ subgroups of $\mathbb{T}$.*

Proof. Since the classes of functions in this proposition are translation invariant groups, the group property of the sets follows from Lemma 1.4.

Thus it is enough to prove that for any $f \in L_0$ the set

$$H = \left\{ h : \|\Delta_h f\|_p \leq K \right\}$$

is closed for any $0 < p \leq \infty$.

Suppose that $h_n \in H$ and $h_n \to h$. By Lemma 4.1, we can choose a subsequence (h_{n_k}) such that $f(x + h_{n_k}) \to f(x + h)$ for a. e. $x \in T$. Then clearly also $\Delta_{h_{n_k}} f \to \Delta_h f$ a.e..

If $p = \infty$ then $h_{n_k} \in H$ means that $|\Delta_{h_{n_k}} f| \leq K$ a.e., thus also $|\Delta_h f| \leq K$ a.e. which means that $h \in H$. If $p < \infty$ then $h_{n_k} \in H$ means that $\int |\Delta_{h_{n_k}} f|^p \leq K^p$, so by the Fatou lemma also $\int |\Delta_h f|^p \leq K^p$ which means that $h \in H$. □

Proposition 4.3 *If $f : \mathbb{T} \to \mathbb{R}$ is measurable and $\Delta_h f$ is essentially bounded for each $h \in \mathbb{T}$ then f is also essentially bounded. (That is, $\mathbb{T} \notin \mathfrak{H}(L_0, L_\infty)$.)*

Proof. Let

$$H_n = \Big\{ h : |\Delta_h f| \leq n \ \text{ a. e.} \Big\}.$$

Since f is measurable, H_n is also measurable, so $\cup H_n = \mathbb{T}$ implies that there exists an n such that H_n has positive measure. Then, by a theorem of Steinhaus, the set $H_n + H_n$ contains a neighborhood of 0. Thus kH_n contains the whole $\mathbb{T}$ if k is big enough. Hence, for any $h \in \mathbb{T}$, $|\Delta_h f| \leq kn$ a. e.

Therefore, denoting kn by K, we get that

$$\Big\{ (x, h) : x, h \in T, |f(x + h) - f(x)| > K \Big\}$$

is a measurable subset of $\mathbb{T} \times \mathbb{T}$ and each of its horizontal section is a null-set. Thus, by Fubini's theorem, almost each of its vertical section is also a null-set, which means that for almost any $x \in T$, $|f(x + h) - f(x)| \leq K$ for almost every h. Therefore, choosing a proper x_0, $|f(x)| \leq |f(x_0)| + K$ for almost every x, which means that f is essentially bounded. □

Corollary 4.4 *The class $L_\infty(\mathbb{T})$ has the weak difference property.*

Proof. This is trivial from Lemma 2.12 and Proposition 4.3. □

Proposition 4.5 *If $0 < p < \infty$, $f : \mathbb{T} \to \mathbb{R}$ is measurable, and $\Delta_h f \in L_p$ for each $h \in \mathbb{T}$ then also $f \in L_p$. (That is, $\mathbb{T} \notin \mathfrak{H}(L_0, L_p)$ for $0 < p < \infty$.)*

Proof. M. Laczkovich proved in [12] that the class L_p has the weak difference property for any $0 < p < \infty$, which means that if $\Delta_h f \in L_p$ for each $h \in \mathbb{T}$ then $f = g + H + S$ where $g \in L_p$, H is additive and for any h we have $\Delta_h S = 0$ a.e. Thus $\Delta_h(f - g)$ is constant almost everywhere for any h, so it is essentially continuous for any h. Since $f - g$ is measurable, by Theorem 2.9, $f - g$ is also essentially continuous, which implies that $f = g + (f - g)$ is in L_p. □

From the last three propositions we get the following:

Theorem 4.6 *For any* $0 < p \leq \infty$*,*

$$\begin{aligned} \mathfrak{H}^0(L_0, L_p) &\subset \Big\{\text{the proper } F_\sigma \text{ subgroups of } \mathbb{T}\Big\}, \\ \mathfrak{H}(L_0, L_p) &\subset \mathfrak{F}_\sigma. \ \square \end{aligned}$$

Now, applying the Triangle-inequality lemma (Lemma 1.6), we can prove easily the following two theorems combining Theorem 4.6 with results of the previous sections.

Theorem 4.7

$$\mathfrak{H}(L_0, \mathcal{C}^*) \subset \mathfrak{F}_\sigma.$$

Proof. By the Triangle-inequality lemma

$$\mathfrak{H}(L_0, \mathcal{C}^*) \subset \mathfrak{H}(L_0, L_\infty) \cup \mathfrak{H}(L_\infty, \mathcal{C}^*).$$

By Theorem 4.6 we have $\mathfrak{H}(L_0, L_\infty) \subset \mathfrak{F}_\sigma$, by Corollary 2.17 we have $\mathfrak{H}(L_\infty, \mathcal{C}^*) = \{\text{finite subsets of } \mathbb{T} \cap \mathbb{Q}\} \subset \mathfrak{F}_\sigma$, so we completed the proof. □

Theorem 4.8

$$\mathfrak{H}(L_0, ACF^*) \subset \mathfrak{F}_\sigma.$$

Proof. By the Triangle-inequality lemma we have $\mathfrak{H}(L_0, ACF^*) \subset \mathfrak{H}(L_0, L_1) \cup \mathfrak{H}(L_1, ACF^*)$. By Theorem 4.6, $\mathfrak{H}(L_0, L_1) \subset \mathfrak{F}_\sigma$; by Theorem 3.7, $\mathfrak{H}(L_1, ACF^*) \subset \mathfrak{N} \subset \mathfrak{F}_\sigma$, so we completed the proof. □

Theorem 4.9 *If* $0 < \alpha \leq 1$ *then*

$$\mathfrak{H}(L_0, (\mathrm{Lip}^\alpha)^*) \subset \mathfrak{F}_\sigma.$$

Proof. M. Balcerzak, Z. Buczolich and M. Laczkovich proved in [1] that $\mathfrak{H}(\mathcal{C}, \mathrm{Lip}^\alpha) \subset \mathfrak{F}_\sigma$ (Theorem 1.4). (Actually, they stated it only for $\alpha = 1$ but their proof works without any modification for Lip^α functions as well.) Then, by Proposition 2.3, we have $\mathfrak{H}(\mathcal{C}^*, (\mathrm{Lip}^\alpha)^*) \subset \mathfrak{F}_\sigma$. Then, using the Monotonicity Lemma, the Triangle-inequality lemma and Theorem 4.7, we get

$$\mathfrak{H}(L_0, (\mathrm{Lip}^\alpha)^*) \subset \mathfrak{H}(L_0, \mathcal{C}^*) \cup \mathfrak{H}(\mathcal{C}^*, (\mathrm{Lip}^\alpha)^*) \subset \mathfrak{F}_\sigma. \quad \square$$

Now we can summarize our results:

Theorem 4.10 *If* $L_0 \supset \mathcal{F} \supset \mathcal{G}$ *and* $\mathcal{G}$ *is any of the classes* L_p $(0 < p \leq \infty)$, $\mathcal{C}^*, ACF^*$ *or* $(\mathrm{Lip}^\alpha)^*$ $(0 < \alpha \leq 1)$ *then*

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{F}_\sigma.$$

If $\mathcal{C} \supset \mathcal{F} \supset \mathcal{G}$ *and* $\mathcal{G}$ *is any of the classes* ACF *or* Lip^α $(0 < \alpha \leq 1)$ *then also*

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{F}_\sigma.$$

Proof. This follows from Theorems 4.6 – 4.9 using the Monotonicity Lemma and Proposition 2.3. $\square$

5 Functions with L∞, and with Lip^1 differences. The construction of Balcerzak, Buczolich and Laczkovich

In this section we prove that $\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \mathfrak{F}_\sigma$ if $\mathcal{G}$ is either L_∞, $(\mathrm{Lip}^1)^*$ or Lip^1, and $\mathcal{F}$ is a reasonable class of functions.

We proved the inclusion $\mathfrak{H}(\mathcal{F}, \mathcal{G}) \subset \mathfrak{F}_\sigma$ in Section 4 (for these cases), so now we need to prove the other inclusion. That is, for any set $H \in \mathfrak{F}_\sigma$ we need to construct a suitable function. We follow the construction of M. Balcerzak, Z. Buczolich and M. Laczkovich ([1]).

Theorem 5.1

$$\mathfrak{H}\left(\bigcap_{0<p<\infty} L_p,\ L_\infty\right) \supset \mathfrak{F}_\sigma.$$

$$\mathfrak{H}\left(\bigcap_{0<\alpha<1}\mathrm{Lip}^\alpha,\ \mathrm{Lip}^1\right) \supset \mathfrak{F}_\sigma.$$

Proof. For any $A \in \mathfrak{F}_\sigma$ we need a function g_1 and a function f such that

$$g_1 \in \left(\bigcap_{0<p<\infty} L_p\right) \setminus L_\infty \qquad \text{and} \qquad \Delta_h g_1 \in L_\infty \quad \text{for any } h \in A; \quad \text{and}$$

$$f \in \left(\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha\right) \setminus \mathrm{Lip}^1 \qquad \text{and} \qquad \Delta_h f \in \mathrm{Lip}^1 \quad \text{for any } h \in A.$$

It is proved in [1] (in the second part of the proof of Theorem 1.4) that for any $A \in \mathfrak{F}_\sigma$ there exists an infinite nowhere dense closed set B such that kB is also nowhere dense for any $k \in \mathbb{N}$, $B = -B$ and the subgroup of $\mathbb{T}$ generated by B covers A. (We use the following notation: $A + B = \{a + b : a \in A, b \in B\}$. The sets $A - B$ and $-A$ are defined similarly. If $k \in \mathbb{N}$, the k-fold sum $A + \ldots + A$ is denoted by kA.)

Thus we can assume that A is an infinite nowhere dense closed set such that kA is also nowhere dense for any $k \in \mathbb{N}$ and $A = -A$. For any such A, Balcerzak, Buczolich and Laczkovich ([1], proof of the (i)$\Rightarrow$(ii) part of Theorem 1.1) constructed functions g_1 and f with the required properties.

(They proved only that $g_1 \in L_1 \setminus L_\infty$ and $\Delta_h g_1 \in L_\infty$ for any $h \in A$; and that $f \in \mathcal{C} \setminus \mathrm{Lip}^1$ and $\Delta_h f \in \mathrm{Lip}^1$ for any $h \in A$. But, since by construction the range of g_1 is $\{0, 1, 2, \ldots\}$ and the measure of $g_1^{-1}(\{k\})$ is at most $1/(k2^k)$, it is also clear that $g_1 \in L_p$ for any $0 < p < \infty$. And, since $f(x) = \int_0^x (g_1(t) - c)dt$ (where $c = \int_{\mathbb{T}} g_1$), this implies that $f \in \mathrm{Lip}^\alpha$ for any $0 < \alpha < 1$.) □

Now we can determine the classes of sets of form $\mathfrak{H}(\mathcal{F}, L_\infty)$ and $\mathfrak{H}(\mathcal{F}, \mathrm{Lip}^1)$ for any reasonable $\mathcal{F}$.

Theorem 5.2 *If* $\bigcap_{0<p<\infty} L_p \subset \mathcal{F} \subset L_0$ *then*

$$\mathfrak{H}(\mathcal{F}, L_\infty) = \mathfrak{F}_\sigma.$$

In particular for any $0 < p < \infty$

$$\mathfrak{H}(L_p, L_\infty) = \mathfrak{F}_\sigma.$$

Proof. This is trivial from Theorem 4.6 and Theorem 5.1 by the Monotonicity Lemma. □

Theorem 5.3 *If* $\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha \subset \mathcal{F} \subset L_0$ *then*

$$\mathfrak{H}(\mathcal{F}^*, (\mathrm{Lip}^1)^*) = \mathfrak{F}_\sigma.$$

If $\bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha \subset \mathcal{F} \subset \mathcal{C}$ *then*

$$\mathfrak{H}(\mathcal{F}, \mathrm{Lip}^1) = \mathfrak{F}_\sigma.$$

In particular for any $0 < \alpha < 1$

$$\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^1) = \mathfrak{F}_\sigma.$$

Proof. The first equality follows from Theorem 4.9 and Theorem 5.1 by the Monotonicity Lemma. Then the second equality follows from the first one by Proposition 2.3. □

6 Summary

For the classes $L_0 \supset L_p \supset L_\infty \supset \mathcal{C} \supset ACF \supset \mathrm{Lip}^1$ (where $1 \leq p < \infty$) as $\mathcal{F}$ and $\mathcal{G}$, the following table shows our results concerning $\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$.

By the Monotonicity Lemma, each column is a sequence of families of sets decreasing monotonically from the top.

$$\mathfrak{H}(\mathcal{F}^*, \mathcal{G}^*)$$

	L_0	L_p	L_∞	$\mathcal{C}$	ACF	Lip^1
L_0	$*$	$\subset \mathfrak{F}_\sigma$	$\subset \mathfrak{F}_\sigma$	$\subset \mathfrak{F}_\sigma$	$\subset \mathfrak{F}_\sigma$	$\mathfrak{F}_\sigma$
L_p		$*$	$\mathfrak{F}_\sigma$	$\supset \mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{F}_\sigma$
L_∞			$*$	finite subsets of $\mathbb{T} \cap \mathbb{Q}$		$\mathfrak{F}_\sigma$
$\mathcal{C}$				$*$		$\mathfrak{F}_\sigma$
ACF					$*$	$\mathfrak{F}_\sigma$
Lip^1						$*$

Remark 6.1 It is also proved in the author's PhD thesis ([9]) that $\mathfrak{H}(L_p, L_q) \supset \mathfrak{p}\mathfrak{D}$ for any $0 < p < q < \infty$; $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta) \supset \mathfrak{p}\mathfrak{D}$ for any $0 < \alpha < \beta < 1$ and all these $\mathfrak{H}(\mathrm{Lip}^\alpha, \mathrm{Lip}^\beta)$ classes are the same. These results are published elsewhere ([11], [10]).

References

[1] M. Balcerzak, Z. Buczolich and M. Laczkovich, Lipschitz differences and Lipschitz functions, *Colloq. Math.*, **72** (1997), 319-324.

[2] N. K. Bary, *Trigonometric series* (in Russian), Moscow, 1961; English translation: *A Treatise on Trigonometric Series*, Macmillan, New York, 1964.

[3] N. G. de Bruijn, Functions whose differences belong to a given class, *Nieuw Arch. Wisk.* **23** (1951), 194-218.

[4] N. G. de Bruijn, A difference property for Riemann integrable functions and for some similar classes of functions, *Indag. Math.* **14** (1952), 145-151.

[5] L. Bukovský, N. N. Kholshchevnikova and M. Repický, Thin sets of harmonic analysis and infinite combinatorics, *Real Anal. Exchange*, **20** (1994-1995), 454-509.

[6] B. Host, J-F. Méla and F. Parreau, Non singular transformations and spectral analysis of measures, *Bull. Soc. Math. France*, **119** (1991), 33-90.

[7] S. Kahane, Antistable classes of thin sets in harmonic analysis, *Illinois J. Math.* **37** (1993), 186-223.

[8] T. Keleti, On the differences and sums of periodic measurable functions, *Acta Math. Hungar.*, **75 (4)** (1997), 279-286.

[9] T. Keleti, *Difference functions of periodic measurable functions*, PhD thesis, Eötvös Loránd University, Budapest, 1996 (http://www.cs.elte.hu/phd_th/).

[10] T. Keleti, Periodic Lip$^\alpha$ functions with Lip$^\beta$ difference functions, to appear in *Colloq. Math.*

[11] T. Keleti, Periodic L_p functions with L_q difference functions, to appear in *Real. Anal. Exchange.*

[12] M. Laczkovich, Functions with measurable differences, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 217-235.

[13] M. Laczkovich, On the difference property of the class of pointwise discontinuous functions and some related classes, *Canad. J. Math.*, **36** (1984), 756-768.

[14] M. Laczkovich and Sz. Révész, Periodic decompositions of continuous functions, *Acta Math. Hungar.*, **54 (3-4)** (1989), 329-341.

[15] M. Laczkovich and I. Z. Ruzsa, Measure of sumsets and ejective sets I., *Real Anal. Exchange*, **22 (1)** (1996-97), 153-166.

[16] W. Sierpiński, Sur les translations des ensembles linéaires, *Fund. Math.*, **19** (1932), 22-28.

[17] A. Zygmund, *Trigonometric series* Vol. I-II, Cambridge University Press, 1959.

DEPARTMENT OF ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
1088 BUDAPEST, MÚZEUM KRT. 6-8, HUNGARY
e-mail: elek@cs.elte.hu