On a Generalization of the Koch Curve Built from n-gons

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Abstract

We study a generalization of the von Koch Curve, which has two parameters, an integer n and a real number c on the interval $(0, 1)$. This von Koch type curve is constructed as the limit of a recursive process that starts with a regular n-gon (or line segment) and repeatedly replaces the middle c portion of an interval by the $n-1$ other sides of a regular n-gon placed contiguous to the interval. We show that there are values of n such that the set of c for which the (n, c)-von Koch Curve is simple, i.e. does not intersect itself, is not an interval.

1 Introduction

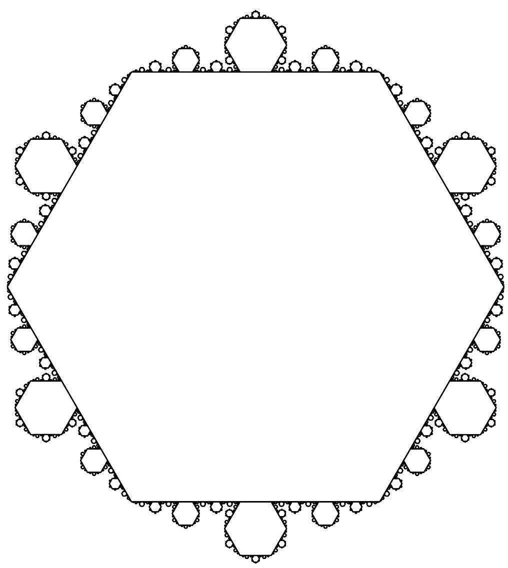

Figure 1: The von Koch "Snowflake" for $n = 6$, $c = 0.125$

Consider the following generalization of the classical triadic von Koch snowflake. Let $0 < c < 1$ and an integer $n \geq 3$ be given. Starting from a regular n-gon, repeatedly replace the middle portion c of each interval by the $n-1$ other sides of a regular n-gon placed contiguous to the interval (see Figure 1.) The limit curve is the (n, c)-*snowflake curve*. The closure of the union of all the above regular n-gons is the (n, c)-*snowflake domain*. The curve we get if we start from a segment is the (n, c)-*von Koch curve*.

In [3] the special case $n = 3, c \leq \frac{1}{3}$ is called "modified von Koch curve." A specific $(3, c)$-snowflake curve with c slightly less than $\frac{1}{2}$ appears in [5, Plate 56]. Note that $n = 3, c = \frac{1}{3}$ gives the

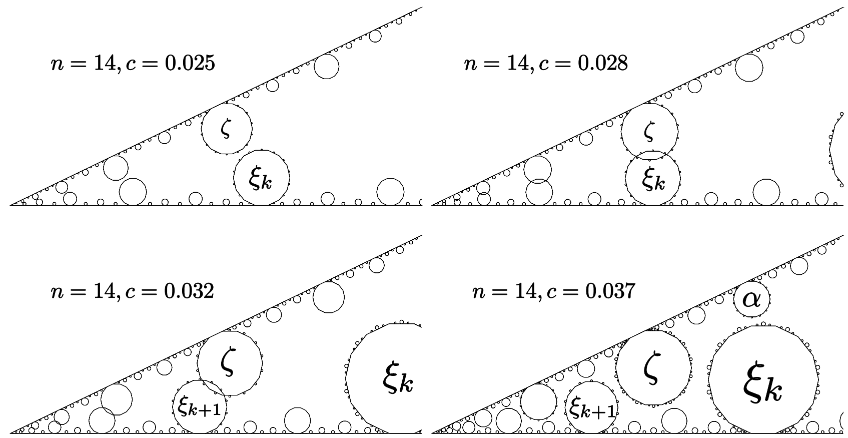

Figure 2: These figures demonstrate the meshing phenomenon.

classical case. Note also that the (n, c)-von Koch curve is a self-similar set and the (n, c)-snowflake consists of n copies of the (n, c)-von Koch curve.

There are two ways to relate the self-intersection of these various constructions (which are elementary to prove). First, if the (n, c)-snowflake curve is a simple curve (in other words if it is *self-avoiding / non-self-intersecting*) then it is the boundary of the (n, c)-snowflake domain. Second, the (n, c)-snowflake curve is self-intersecting if and only if the (n, c)-von Koch curve is self-intersecting.

M. van den Berg studied the heat equation for planar regions similar to the (n, c)-snowflake domain (see e.g. [1, 2]). He noticed that the $(4, c)$-snowflake curve is self-avoiding if and only if $c < \frac{1}{3}$ and asked for analogous result for the $(3, c)$-snowflake. In [4] it was proved that the $(3, c)$-snowflake curve is non-self-intersecting if and only if $c < \frac{1}{2}$. Then, M. van den Berg asked what the critical c is for other n.

The critical c phenomenon is a peculiarity of the low order cases, where a certain symmetry effects a self-intersection. A shorter proof of the result that the $(3, c)$-snowflake curve self-intersects for $c \geq \frac{1}{2}$ is presented to illustrate this symmetry.

However, the main goal of the present paper is showing that in general there is no such critical c: for some n there exists $c_1 < c_2$ such that the (n, c_1)-snowflake curve is self-intersecting but the (n, c_2)-snowflake curve is not (Theorem 4.15).

Figure 2 shows how this phenomenon can happen. Table 3 shows a list of triplets (n, c_1, c_2) with the above property.

2 Terminology

The starting segment of an (n, c)-von Koch curve is the *base* of the curves.

By self-similarity an (n, c)-von Koch curve consists of smaller (n, c)-von Koch curves. We make use of the language of genealogy in describing these von Koch Curves. An (n, c)-von Koch Curve is said to have $n + 1$ children. Two lie on either off-center $(1 - c)/2$ segment. The other $n - 1$ have bases that, together with the center c segment of its parent, form a regular n-gon. These $n - 1$ children are *primary*. Naturally, a von Koch Curve may be described as the union of its children. By *descendants*, we refer to the set curves comprised of a curve's children, its grandchildren, its great grandchildren, etc.

In the same way, this terminology describes the piecewise-linear curves that are used to construct a von Koch Curve. Namely, a line segment has $n + 1$ children, the $n - 1$ primary children that form a *primary n-gon* and two non primary, off-center children.

Here, we define the vertices of an (n, c)-von Koch Curve which we employ in our proof (see Figure 3). The (n, c)-von Koch Curve AC is built upon the closed horizontal line segment AC with length 1 and with C on the right. Unless otherwise specified, all references to an (n, c)-von Koch Curve refer to curve AC. Call M the vertex of the primary n-gon touching the right non primary child. Call S the vertex of the primary n-gon adjacent to M on the right, and lastly call T the vertex adjacent to S on the right.

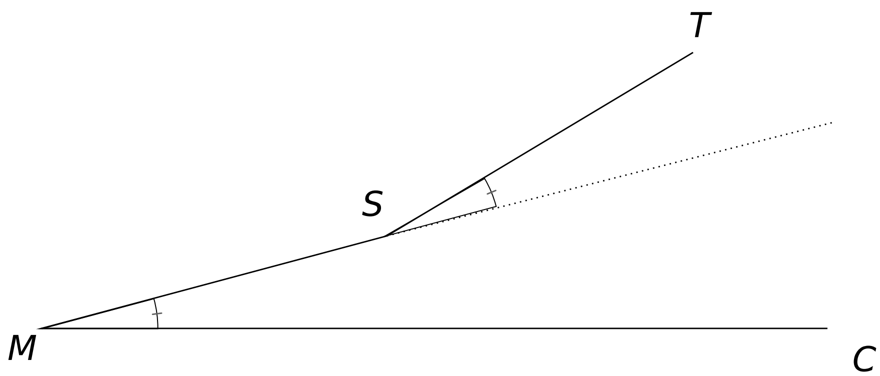

Figure 3: The wedge-shaped region between segments MS and MC is singularly important in the search for self-intersection.

We employ an orthonormal coordinate system centered at M. The first coordinate extends along ray MC. The second coordinate, naturally, extends upwards. Distance is Euclidean, and again line segment AC has length 1.

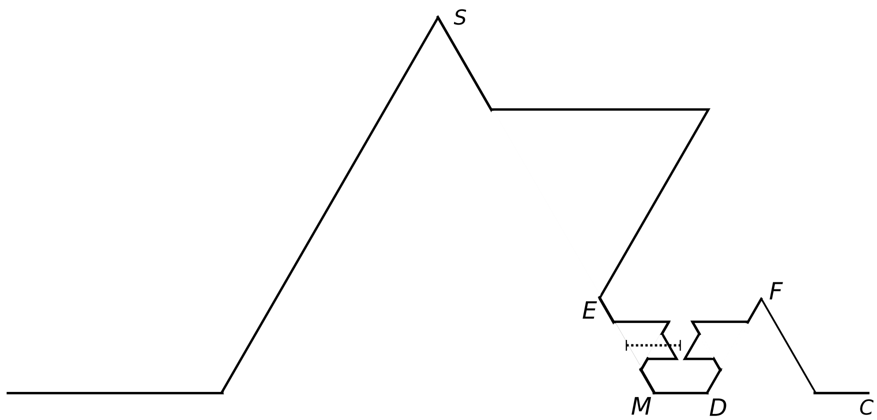

Figure 4: The value of c is $\frac{1}{2}$. The dotted line segment is contained in the closure of the children of curve MS. This is sufficiently long to effect self-intersection.

3 Low Order Symmetry

In [4] Theorem 3.2 and its converse were proved. The proof of Theorem 3.2 presented here is substantially simpler than the proof in [4], where the converse has a simple proof.

Remark 3.1. The proof of the converse of Theorem 3.2 in [4] can be generalized to show that any (n, c)-snowflake is self-avoiding in an interval of c around 0.

Theorem 3.2. *The $(3, c)$-snowflake curve self-intersects when $c \geq \frac{1}{2}$.*

Proof. The proof can be quite simply made once a few definitions are made. Let DF be the inward facing primary child of curve MC. Let EM be the lower non primary child of SM. Figure 4 illustrates these.

The key realization is that DF and EM are the same size for any c. Thus, consider a lineage of triangles emanating from EM towards DF, which alternates between right and left primary children (so to become as close as possible to the midpoint of DF). In the same fashion, consider the symmetric (about the planar reflection that sends D to M) lineage of triangles coming from DF to EM. A quick calculation shows that these two sequences of triangles overlap when $c \geq \frac{1}{2}$.

The horizontal distance separating the midpoints of EM and DF is calculated to be

$$\left(\frac{1-c}{2}\right)^2 + 2\frac{c(1-c)}{8} = \frac{1}{4} - \frac{c}{4}.$$

The horizontal length of *each* lineage, taking the midpoints of EM and DF to be the starting points, is

$$\frac{c^2(1-c)}{4}\left(1 + c + c^2 + c^3 + \cdots\right) = \frac{c^2}{4}.$$

This leads to the inequality

$$\frac{c^2}{2} + \frac{c}{4} \geq \frac{1}{4}.$$

This inequality is satisfied so long as $c \geq \frac{1}{2}$, establishing that the two sequences of triangles must overlap, and so the curve self-intersects. □

An analogous result can be formulated for the $(4, c)$-von Koch Curve using essentially the same method of proof. In larger n-gons, the lineages can still be used to provide a value of c above which the curve certainly self-intersects. However, the curve can self-intersect long before the symmetric lineages overlap, making the lineages somewhat irrelevant. For larger n-gons, there is an interesting *mesoscopic* region of c, below which the curve is certainly self-avoiding and above which there is certain self-intersection. The self-intersection pattern in this mesoscopic region is quite complex.

4 Main Result

We show that the set of c for which the (n, c)-von Koch Curve self-intersects is not always interval.

It is superfluous to search for intersections everywhere on a curve AC. Instead, a wedge-shaped region between segments MS and MC is all that needs to be inspected. This region is illustrated in Figures 3 and 5. We must expound that by self-similarity, it is superfluous to check for intersections between descendants of curve MS with other descendants of curve MS (and likewise between descendants of curve MC with other descendants of curve MC), as this is equivalent to searching for self-intersections of curve AC.

In the wedge shaped region between segments MS and MC, there is a considerable amount of repetition. In order to simplify the search for self-intersection, this repetition needs to be eliminated, and so we introduce the following map.

Definition 4.1. The *rescaling map* is defined as $\vec{x} \mapsto \vec{x}(1-c)/2$. The rescaling map sends curve MS into itself, and it sends curve MC into itself. The *rescaled image* is the image of curve AC under the rescaling map.

Lemma 4.2. *Curve AC self-intersects if and only if there is an intersection between curve MS and curve MC.*

Proof. Suppose the curve MC is not strictly between rays MS and MC. This implies that curve MC must intersect ray MS at some point besides M. Iterated applications of the rescaling map bring this point arbitrarily close to M. Hence there is a curve descended from curve MC having endpoint M whose other endpoint is arbitrarily close to M, and so it can be ensured that part of curve MC crosses line segment MS. As a result, there must be an intersection between curves MS and MC.

Suppose the curve MC is strictly between rays MS and MC. By similarity, curve ST is strictly between rays ST and MS, and thus curve ST is separated from MC. All other children of curve AC are strictly separated from curve MC by line MS. Furthermore, all primary children of curve AC are pairwise disjoint, and thus if an intersection occurs, it must be between curve MS and curve MC (or between the left counterparts of curve MS and MC, but by the natural left-right symmetry of curve AC, it is unnecessary to search both the left and right sides for intersection).

□

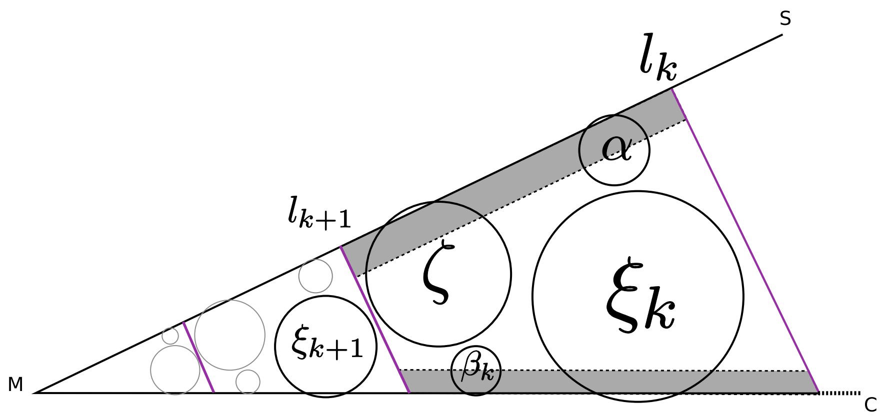

Figure 5: $n = 14$, $c = 0.037$, with circular approximations. The grayed circles do not need to be checked by self-similarity.

Using the rescaling map, Lemma 4.2 can be bettered. Only a properly chosen trapezoid needs to be checked, since by iterated applications of the rescaling map, the whole wedge is covered.

In order to isolate parts of the snowflake domain from others, we circumscribe a circle around the $n-1$ primary children of a curve. For a curve of base length 1, we define $r(c)$ to be the radius of the smallest circle containing (not strictly) all this curve's primary children. In this way, a curve gives rise to exactly one circle.

Only some of the circles are relevant to showing self-avoidance, and these are introduced here.

Definition 4.3. The relevant circles are the following (see Figure 5).

- ζ is the circle associated to curve MS
- α is the largest circle arising between ζ and S
- ξ_n is a family of circles defined recursively. ξ_1 is the circle associated to curve MC, and ξ_{n+1} is the rescaled image of ξ_n.
- β_n is a family of circles called the *first order intermediaries*. Each β_n is defined as the largest circle between ξ_n and ξ_{n+1}.

Only a few of the circles from the two infinite families listed are needed. We limit our search for intersections between ζ, α, and the two families. Only the circles from these families that are closest to ζ and α are relevant to the search, and they are contiguous members of the sequence. This motivates the following definition.

Definition 4.4. Let k be the natural number such that the horizontal coordinate of the center of ζ is between the horizontal coordinates of the centers ξ_k and ξ_{k+1}.

We are now able to define the critical trapezoid in which intersections are sought. The trapezoid is the region of the plane bounded by line segments MS and MC, between two parallel lines l_k and l_{k+1}, which are defined as follows.

Definition 4.5. Let $\{l_m\}$, with $m \geq k$, be a family of line segments, defined recursively with l_{m+1} being the rescaled image of l_m. Furthermore, l_k should be chosen so that l_{k+1} is a line segment, which:

- begins on segment MS and ends on segment MC
- if extended to a line, strictly separates ξ_{k+1} and the rescaled image of α from ζ and β_k.

Remark 4.6. These line segments do not exist for all c, and so for every c for which they are used, it must be shown first that they exist.

Definition 4.7. Let the *critical trapezoid* be the region of the plane bounded by l_k, l_{k+1}, segment MS, and segment MC.

Whenever the critical trapezoid exists, there are 4 tractable circles inside of it. However, these circles do not contain all portions of the curves MS and MC inside the critical trapezoid. There are two final regions that need to be defined to account for all parts of the curve between MS and MC. These are the upper and lower strips.

Definition 4.8. Let the *upper strip* be the minimum constant-radius tubular neighborhood of segment MS containing those descendants of curve MS within the critical trapezoid that are not contained in α or ζ. Analogously let the *lower strip* be the minimum constant-radius tubular neighborhood of segment MC containing those descendants of curve MC within the critical trapezoid that are not contained in either ξ_k or β_k. We designate μ to be the radius of the upper strip and λ to be the radius of the lower strip.

Lemma 4.9 collects the importance of the preceding definitions and remarks.

Lemma 4.9. *If for a given c, the line segments $\{l_m\}$ exist, and the upper objects are disjoint from the lower, (i.e. α, ζ, and the upper strip are disjoint from ξ_k, β_k, the lower strip) then curve AC does not self-intersect.*

Proof. Because the line segments l_k and l_{k+1} exist, the critical trapezoid, denoted T, exists. Because the rescaling map takes l_m to l_{m+1} for any $m \geq k+1$, the iterated rescaled images of the T cover the entire wedge between M and l_m. There is a portion of the wedge to the right of l_m not covered by these iterated rescaled images. Once the self-avoidance of the wedge portion between M and l_m is established, the rest of the wedge can be covered by the inverse rescaled image applied to wedge between M and l_m. Thus, it suffices to check T for intersections, and by the definition of the upper and lower strips, curve MS is disjoint from curve MC inside T. □

Thus, once the existence of l_k has been shown, it is straightforward to show that the curve is self-avoiding. One sufficient and computationally manageable criterion is the following.

Definition 4.10. Define $\vec{v}$ to be the unit vector pointing along segment MS.

Lemma 4.11. *Define a linear functional v^* by $\vec{x} \mapsto \vec{x} \cdot \vec{v}$; then, denoting by $\tilde{\alpha}$ the rescaled image of α, the line segment l_k exists if*

$$\sup_{x \in \tilde{\alpha}} v^*(x) \leq \sup_{x \in \xi_{k+1}} v^*(x) < \inf_{x \in \zeta} v^*(x) \leq \inf_{x \in \beta_k} v^*(x).$$

Proof. The functional v^* is in fact a semi-norm on the wedge shaped area. Thus, choose some y between $\sup_{x \in \xi_{k+1}} v^*(x)$ and $\inf_{x \in \zeta} v*(x)$, and let l_{k+1} be defined as the preimage of y under the restriction of v^* to the wedge. □

Remark 4.12. This choice of $\vec{v}$ is arbitrary, based on what seems to work in some cases. Any unit vector between segments MS and MC would do.

All of these lemmas together give a method for checking that the curve is self-avoiding. On the other hand, there also needs to be a method for checking if the curve is self-intersecting.

The base segments of the primary children of a curve form all but one sides of a regular n-gon. Inside of this n-gon, inscribe a circle. This inscribed circle is concentric with the outer approximation constructed earlier.

Definition 4.13. Let $\rho(c)$ be the radius of the circle inscribed inside of the n-gon formed by the bases of the primary children of a curve with unit base length.

Lemma 4.14. *If the inscribed circles associated to two curves overlap, then the von Koch Curve self-intersects.*

Proof. In particular, if the inscribed circles overlap, then the domains that the two curves bound overlap. Thus, the Snowflake domain overlaps, which is equivalent to the von Koch Curve self-intersecting. □

We now have the machinery to prove the main result.

Table 1: These values establish the existence of the $\{l_m\}$.

$\sup_{x\in\tilde{\alpha}} v^*(x)$	$\sup_{x\in\xi_{k+1}} v^*(x)$	$\inf_{x\in\zeta} v^*(x)$	$\inf_{x\in\beta_k} v^*(x)$
0.014275	0.014828	0.015270	0.016623

Table 2: Here, μ refers to the upper strip, and λ refers to the lower strip.

	β_4	ξ_4	λ
α	0.009329	0.000352	0.008846
ζ	0.000162	0.001038	0.001609
μ	0.005625	0.002349	0.006012

Theorem 4.15. *The set of c for which the (n, c)-von Koch Curve is self-avoiding is not in general an interval.*

Proof. We present numerical proof that when n is 14, the set of self-avoiding c is not an interval. In particular, when $c = 0.037$, the curve is self-avoiding, but when $c = 0.032$, the curve self-intersects (see Figure 2). It is assumed clear that for sufficiently small c ($c \ll 0.032$), the curve is self-avoiding.

For both of these values of c, the value of k is computed to be 4. When $c = 0.037$, Lemma 4.9 holds that the curve is self-avoiding, so long as the critical trapezoid exists. Figure 1 shows that the criteria of Lemma 4.11 are satisfied, and hence the critical trapezoid exists.

The upper objects are shown to be separated from the lower objects. Table 2 shows the minimal distances between each of the objects. Formulae to compute these results are provided in the Appendix. Note that the minimal distance between the upper strip and lower strip is along the line l_{k+1}. For the value in Table 2, the line l_{k+1} is chosen so that it bisects the v^*-distance between ξ_{k+1} and ζ.

As the upper objects are disjoint from the lower objects, with a separating distance that is well beyond numerical error, we conclude that at $c = 0.037$, the curve is self-avoiding.

To show that the curve self-intersects when $c = 0.032$, we inscribe a circle inside of the n-gon as remarked previously. The radius of the inscribed circle of curve AC is

$$\frac{c}{2}\tan(\theta/2). \tag{1}$$

We can then compute the distance between the centers of ζ and ξ_5 minus the sum of inscribed radii of ζ and ξ_5

$$\approx -0.0003895. \tag{2}$$

And so by Lemma 4.14 von Koch Curve certainly self-intersects when $c = 0.032$. □

Table 3: These are values of n up to 50 where the result given here may be repeated using precisely the same methodology. At c_1 the curve can be shown to self-intersect, and at c_2 the curve can be shown to be self-avoiding.

n	c_1	c_2
14	0.032	0.037
19	0.014670	0.018424
20	0.014571	0.018028
26	0.0074988	0.0092013
27	0.0074905	0.0091380
28	0.0074653	0.0090610
29	0.0074555	0.0089734
30	0.0074584	0.0089705
31	0.0074459	0.0088527
37	0.0038070	0.0044192
38	0.0037970	0.0044116
39	0.0037992	0.0043830
40	0.0037966	0.0043671
41	0.0038046	0.0043477
42	0.0037935	0.0043392
43	0.0037928	0.0043135
44	0.0037883	0.0042984

5 Conclusion

The methodology used here also may be used to show that the set of self-avoiding c is not an interval for higher n. In Table 3, some values of n, c_1, and c_2 with $c_1 < c_2$ are given such that the (n, c)-von Koch Curve can be shown to self-intersect at c_1 and can be shown to be self-avoiding at c_2 using the machinery developed here.

While the set of self-avoiding c is not necessarily an interval, it is unknown how complex the set might be. For example, a challenging open question is whether or not the set of self-avoiding c can be written as a finite union of intervals, or for that matter a countable collection of intervals (note that generalized Cantor sets appear in various places in the Snowflake domain).

Particularly, for this question, the aid of computer tools is limited, which have been indispensable in studying the von Koch Curve. Java software for exploring the fractal and generating figures (such as Figure 1) is available on request, as is an MPEG animation linking the stills in Figure 2.

Appendix

All the formulae for the computations are presented here. In what follows, the variable θ is the interior angle of a regular n-gon, as in Formula 3.

The radius $r(c)$ of the minimal circle enclosing all of a unit base-length curve's primary children is given by Formula 4. Formula 4 may be derived by analyzing the relation between the radius of the minimal circle enclosing a von Koch Curve's primary child and the minimal circle enclosing a primary child of a von Koch Curve's primary child. The grandchild's circle is tangent to and contained in the child's circle. Thus, by writing the radius of the circle enclosing the grandchild in two ways, the following relation is derived

$$cr(c) = r(c) - \frac{c}{2}\tan(\theta/2)(1+c).$$

This immediately yields the presented formula.

The distance $h(c)$ from the center of the minimal circle to the base segment of a unit base-length curve is given by Formula 5. To derive this, note that this minimal circle is concentric with the center of the polygon formed by this curve's primary children.

Table 4: These are the relevant formulae for the circles shown to be disjoint.

	Horizontal Coordinate	Vertical Coordinate	Radius
ξ_k	$\left(\frac{1-c}{2}\right)^k \frac{1}{2}$	$\left(\frac{1-c}{2}\right)^k h(c)$	$\left(\frac{1-c}{2}\right)^k r(c)$
β_k	$\left(\frac{1-c}{2}\right)^{k+1} \left(\frac{3+c}{4}\right)$	$\left(\frac{1-c}{2}\right)^{k+2} h(c)$	$\left(\frac{1-c}{2}\right)^{k+2} r(c)$
ζ	$-\frac{c}{2}\cos(\theta) + c\sin(\theta)\,h(c)$	$\frac{c}{2}\sin(\theta) + c\cos(\theta)\,h(c)$	$cr(c)$
α	$\frac{-c}{4}(3+c)\cos(\theta)$ $+\frac{c}{2}(1-c)\sin(\theta)\,h(c)$	$\frac{c}{4}(3+c)\sin(\theta)$ $+\frac{c}{2}(1-c)\cos(\theta)\,h(c)$	$c\frac{1-c}{2}r(c)$

Recall that μ and λ are defined to be the radii of the upper and lower strips (see Definition 4.8). These can be computed presupposing the existence of the critical trapezoid, which are given as Equations 7 and 8. To derive these, observe that the approximating circle of a non primary child rises exactly as high as the center of the approximating circle of its parent. On curve MS, the next largest circle inside the critical trapezoid after α is a non primary child of the curve that α approximates. On curve MC, the next largest circle inside the critical trapezoid after β_k is a non primary child of the curve that β_k approximates.

The definition of k holds that k should satisfy the following inequality, which

Table 5: These are the all the formulae, besides the circle formulae in Table 4, required to employ Theorem 4.15.

The interior angle of an n-gon	θ	$= \dfrac{(n-2)\pi}{n}$	(3)
The radius of an approximating circle	$r\,(c)$	$= \dfrac{c}{2}\dfrac{1+c}{1-c}\tan(\theta/2)$	(4)
The height of the center of a circle above its base	$h\,(c)$	$= \dfrac{c}{2}\tan(\theta/2)$	(5)
The index of the ξ immediately to the right of ζ	k	$= \left\lfloor \dfrac{\ln\left[c(2\sin(\theta/2)^2(1+c)-1)\right]}{\ln\left[(1-c)(1/2)\right]} \right\rfloor$	(6)
The thickness of the upper strip	μ	$= c\left(\dfrac{1-c}{2}\right)^2 h(c)$	(7)
The thickness of the lower strip	λ	$= \left(\dfrac{1-c}{2}\right)^{k+3} h(c)$	(8)

can be assembled from the formulae in Tables 4 and 5

$$\left(\frac{1-c}{2}\right)^{k+1} \leq c\big(-\cos\theta + c\tan\left(\frac{\theta}{2}\right)\sin\theta\big) \leq \left(\frac{1-c}{2}\right)^{k}.$$

This inequality may be solved for k, yielding Formula 6.

The circle formulae in Table 4 can be deduced directly from their definitions. The coordinates for the centers of α and ζ are easiest to deduce if they are first computed in the rotated coordinate system where segment MS is rotated to be the negative direction on the horizontal axis.

Acknowledgments

The research of the second author was partially supported by Hungarian Scientific Foundation grant No. T49786. This work began when the second author visited M. van den Berg at the University of Bristol.

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