ON THE DIFFERENCES AND SUMS OF PERIODIC MEASURABLE FUNCTIONS

BY

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1. Introduction

Let $f : \mathbf{R} \to \mathbf{R}$ be a real function and let $\alpha_1, \dots, \alpha_n$ be given real numbers. We say that $f = f_1 + \dots + f_n$ is an $(\alpha_1, \dots, \alpha_n)$-*decomposition* of f if f_i is periodic modulo α_i for every $i = 1, \dots, n$. We say that $f = f_1 + \dots + f_n + p$ is an $(\alpha_1, \dots, \alpha_n)$-*quasi-decomposition* of f if p is a polynomial of degree $< n$ and f_i is periodic modulo α_i for every $i = 1, \dots, n$. (The decomposition or the quasi-decomposition is said to be continuous or measurable if all the functions f_i are continuous or measurable resp.) If f has an $(\alpha_1, \dots, \alpha_n)$-decomposition or an $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition, then it is easy to see that $\Delta_{\alpha_1} \dots \Delta_{\alpha_n} f = 0$, where Δ_α is the difference operator with step α, that is $\Delta_\alpha f(x) = f(x + \alpha) - f(x)$. A class $\mathcal{F}$ of real functions is said to have the *decomposition property* if, for every $f \in \mathcal{F}$ and $\alpha_1, \dots, \alpha_n \in \mathbf{R}$, the equation $\Delta_{\alpha_1} \dots \Delta_{\alpha_n} f = 0$ implies that f has an $(\alpha_1, \dots, \alpha_n)$-decomposition with functions in $\mathcal{F}$.

These notions were introduced in [2] by M. Laczkovich and Sz. Révész. They proved in [2] and [3] that a series of important classes of real functions have the decomposition property (e.g. $L_\infty(\mathbf{R})$, the class of bounded Lipschitz functions and the class of bounded continuous functions). On the other hand it is easy to see that the class of all $\mathbf{R} \to \mathbf{R}$ functions and the class of all continuous functions do not have the decomposition property. ($f(x) = x$ is a counter-example for both classes.) It was asked in [3] whether it is true that if f is continuous and has a measurable $(\alpha_1, \dots, \alpha_n)$-decomposition then f also has a continuous $(\alpha_1, \dots, \alpha_n)$-decomposition or at least a continuous $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition.

If f is uniformly continuous, then it is an easy consequence of a result in [3], that the answer is affirmative. However, as we shall prove below, in the general case the answer is negative: we construct two measurable periodic functions such that their sums are continuous and unbounded.

A set $\Lambda \subset \mathbf{R}$ is called *pseudo-Dirichlet* if there exists a sequence of integers $0 < q_1 < q_2 < \dots$ and a sequence of positive reals $\varepsilon_1, \varepsilon_2, \dots$ converging to zero such that, for any $\alpha \in \Lambda$, if n is big enough then

$$|\sin q_n \pi \alpha| \le \varepsilon_n \qquad \big(n \ge N(x)\big).$$

This notion together with several other notion of thinness in harmonic analysis is discussed in the survey [1] of L. Bukovskỳ, N. N. Kholshchevnikova and M. Repickỳ. It is known, for example, that every countable set and some nonempty perfect sets are pseudo-Dirichlet.

For any pseudo-Dirichlet set $\Lambda \subset \mathbf{R}$ we construct a modulo 1 periodic real function $h \in L_2 \setminus L_\infty$ for which the function $\Delta_\alpha h(x) = h(x + \alpha) - h(x)$ is continuous for any $\alpha \in \Lambda$.

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(Here and in the sequel by L_2 we mean the class of those measurable functions $f : \mathbf{R} \to \mathbf{R}$ for which f^2 has a finite integral on any finite interval. The class L_∞ consists of those real functions that are almost everywhere equal to a bounded function.) Adding a proper modulo α periodic function $g_\alpha(x)$ for each $\alpha \in \Lambda$ to this function h we will get continuous functions $g_\alpha + h$ for all α. As we will see later these sums cannot be bounded. In this way we can construct simultaneously many pairs of measurable periodic functions such that their sum is continuous and unbounded.

2. Some facts

In this section we prove some easy facts and corollaries that do not seem to have been stated previously. We will use the following known (and easy to prove) fact ([3]):

Proposition 0. *No non-constant polynomial (more generally no function f with $\lim_{x\to\infty} |f| = \infty$) can be the sum of finitely many periodic measurable functions.*

Proposition 1. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers such that α_i/α_j is irrational for every $1 \le i < j \le n$. Then a function $f : \mathbf{R} \to \mathbf{R}$ can have at most one measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition, apart from additive constants and changes on a null-set.*

Proof. Clearly it is enough to prove that if

(1) $$f_1 + \ldots + f_n = 0 \qquad \text{almost everywhere,}$$

where f_j is measurable and periodic modulo α_j for all j, then f_n is constant almost everywhere. We prove it by induction. For $n = 1$ it is obvious. Let $n \ge 2$. Since f_1 is periodic modulo α_1 the equation (1) implies

$$\Delta_{\alpha_1} f_2 + \ldots + \Delta_{\alpha_1} f_n = 0 \qquad \text{almost everywhere.}$$

Now using the induction assumption we get that $\Delta_{\alpha_1} f_n = c$ almost everywhere. Then the function $g(x) = f_n(x) - (c/\alpha_1)x$ is measurable and periodic modulo α_1 except a set of measure zero. Thus $(c/\alpha_1)x$ is the sum of two measurable periodic functions, which implies, using Proposition 0, that $c = 0$. Therefore the measurable function f_n has two incommensurable periods (except a set of measure zero), which implies that f_n is constant almost everywhere. This completes the proof.

Proposition 2. *If $f = f_1 + \ldots + f_n$, $f \in L_\infty$, the functions f_i are measurable and periodic modulo α_i $(i = 1, \ldots, n)$, where α_i/α_j is irrational for every $i \ne j$, then the functions $f_1, \ldots, f_n$ are also in L_∞.*

Proof. It is proved in [2] that the class of functions L_∞ has the decomposition property. Using Proposition 1, this implies Proposition 2.

Proposition 3. *If f has a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition and a measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition, then f has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition, too.*

Proof. Suppose that $f_1 + \ldots + f_n = f = g_1 + \ldots + g_n + p$, where the functions f_i are measurable, the functions g_i are continuous and p is a polynomial. Then $p = (f_1 - g_1) + \ldots + (f_n - g_n)$, so using Proposition 0, p must be constant, so $g_1 + \ldots + g_{n-1} + (g_n + p)$ is a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition of f.

Theorem 0. *If a uniformly continuous function f has a measurable $(\alpha_1, \ldots, \alpha_n)$-decomposition, then it also has a continuous $(\alpha_1, \ldots, \alpha_n)$-decomposition.*

Proof. It is proved in [3] (4.2.Thm.) that a function f has a continuous $(\alpha_1, \ldots, \alpha_n)$-quasi-decomposition with linear p if and only if $\Delta_{\alpha_1} \ldots \Delta_{\alpha_n} f = 0$ and f is uniformly continuous. By the previous proposition this implies the theorem.

Remark 1. By Proposition 1, this theorem implies that if the sum of finitely many measurable periodic functions with pairwise incommensurable periods is uniformly continuous, then these functions must be continuous apart from changes on a null-set.

Remark 2. These results do not remain true if we omit the word 'uniformly'. This is one of the main results of this paper. For bounded continuous functions, however, these results become true again, since the class of bounded continuous functions has the decomposition property.

3. Some lemmas

Lemma 1. *If the integers $0 < q_1 < q_2 < \ldots$ satisfy the condition*

$$(*) \qquad q_N \sum_{n=N+1}^{\infty} \frac{1}{n q_n} \le C \qquad (\text{ for a fixed } C \text{ for any } N)$$

then the modulo 1 periodic L_2 function h with Fourier series

$$h(x) \sim \sum_{n=1}^{\infty} \frac{\cos(2\pi q_n x)}{n}$$

cannot be in L_∞.

Proof. For a fixed N we use the following notation:

$$x_N = \frac{1}{6q_N}, \qquad S_N = \sum_{n=1}^{N} \frac{1}{n}, \qquad m_N(x) = h(x) - \sum_{n=1}^{N} \frac{\cos(2\pi q_n x)}{n} \sim \sum_{n=N+1}^{\infty} \frac{\cos(2\pi q_n x)}{n}.$$

Then for any $x \in [0, x_N]$

$$h(x) = \sum_{n=1}^{N} \frac{\cos(2\pi q_n x)}{n} + m_N(x) \ge \frac{1}{2} S_N + m_N(x).$$

On the other hand, using the term by term integrability of Fourier series,

$$\left|\int_0^{x_N} m_N(x)dx\right| = \left|\sum_{n=N+1}^{\infty}\int_0^{x_N}\frac{\cos(2\pi q_n x)}{n}\right| = \left|\sum_{n=N+1}^{\infty}\frac{\sin(2\pi q_n x_N)}{2\pi q_n n}\right| \le \sum_{n=N+1}^{\infty}\frac{1}{6q_n n}.$$

Therefore the average of $m_N(x)$ on the interval $[0, x_N]$ is

$$\left|\frac{\int_0^{x_N} m_N(x)dx}{x_N}\right| \le q_N \sum_{n=N+1}^{\infty}\frac{1}{q_n n} \le C.$$

Thus on a subset H_N of $[0, x_N]$ with positive Lebesgue measure $m_N(x) \ge -C$, which implies that

$$h(x) \ge \frac{1}{2}S_N + m_N(x) \ge \frac{1}{2}S_N - C \qquad (x \in H_N).$$

On the other hand $S_N \to \infty$, which means that $h \notin L_\infty$.

Lemma 2. *Let A be an additive subgroup of $\mathbf{R}$ and let S be a dense union of translated copies of A. Suppose that we have a function $\overline{h} : \mathbf{R} \to \mathbf{R}$ and continuous functions $l_a : \mathbf{R} \to \mathbf{R}$ for all $a \in A$ such that $\Delta_a \overline{h}|S = l_a|S$ for any $a \in A$.*

Then there exists a function $h : \mathbf{R} \to \mathbf{R}$ such that $h|S = \overline{h}|S$ and $\Delta_a h = l_a$ for every $a \in A$.

Proof. Choose a point from all those translated copies of A that is not in S. Denote this set by B. Then every $x \notin S$ can be uniquely written in the form $x = \beta + b$ where $\beta \in B$ and $b \in A$. For such an $x \notin S$ let $h(x) = l_b(\beta)$. For $x \in S$ let $h(x) = \overline{h}(x)$.

Now we need to prove that $\Delta_a h(x) = l_a(x)$ for all $a \in A$ and $x \in \mathbf{R}$. We know that this equality holds if $x \in S$. If $x \notin S$ then $x = \beta + b$ $(\beta \in B,\ b \in A)$, so using the definition of h:

$$\Delta_a h(x) = \Delta_a h(\beta + b) = h(\beta + b + a) - h(\beta + b) = l_{b+a}(\beta) - l_b(\beta).$$

So it is enough to prove that

$$l_a(\beta + b) = l_{b+a}(\beta) - l_b(\beta) \tag{2}$$

for any $\beta \in B$ and $a, b \in A$.

Fix a and b. If β is not in B but in S then (2) holds. Indeed, in this case $\beta + b$ is also in S so $l_a(\beta + b) = \Delta_a \overline{h}(\beta + b)$, $l_{b+a}(\beta) = \Delta_{b+a}\overline{h}(\beta)$ and $l_b(\beta) = \Delta_b \overline{h}(\beta)$ and a trivial calculation shows that $\Delta_a \overline{h}(\beta + b) = \Delta_{b+a}\overline{h}(\beta) - \Delta_b \overline{h}(\beta)$.

Therefore (2) holds on a dense subset of $\mathbf{R}$. On the other hand both sides of this equation are continuous, so this implies that (2) holds on the whole real line, thus specially it also holds on B, which completes the proof of this lemma.

4. The main results

Theorem 1. *For any pseudo-Dirichlet set Λ (e.g. for any countable Λ and also for some nonempty perfect Λ) there exists a modulo 1 periodic function $h \in L_2 \setminus L_\infty$ for which $\Delta_\alpha h$ is continuous for any $\alpha \in \Lambda$.*

Proof. Take a sequence $q_1 < q_2 < \dots$ and a sequence $\varepsilon_n \to 0$ witnessing the pseudo-Dirichlet property of Λ. Then any subsequences of (q_n, ε_n) also witnessing the pseudo-Dirichlet property of Λ, so choosing a subsequence increasing quickly enough in q_n and decreasing quickly enough in ε_n we can assume that (q_n) satisfies the condition $(*)$ of Lemma 1 and also $\varepsilon_n < 1/n^2$. Let A denote the set of all α-s for which there exists a K such that $|\sin q_n \pi \alpha| \le K/n$ for every n. It is easy to check that A is an additive subgroup of $\mathbf{R}$ and $\Lambda \subset A$. Let $\tilde{h} \in L_2$ be a modulo 1 periodic complex valued function with the Fourier series

$$\tilde{h}(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i q_n x}.$$

Let $\overline{h} = \operatorname{Re} \tilde{h}$. Then $\overline{h}(x) \sim \sum_{n=1}^{\infty} \frac{\cos(2\pi q_n x)}{n}$, so according to Lemma 1, $\overline{h} \in L_2 \setminus L_\infty$.

Let $\alpha \in A$ and let K be a real number for which $|\sin q_n \pi \alpha| \le K/n$ $(n = 1, 2, \dots.)$ The Fourier series of $\tilde{h}(x + \alpha)$ is

$$\tilde{h}(x+\alpha) \sim \sum_{n=1}^{\infty} (\frac{1}{n} e^{2\pi i q_n \alpha}) e^{2\pi i q_n x},$$

so

$$\Delta_\alpha \tilde{h}(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} (e^{2\pi i q_n \alpha} - 1) e^{2\pi i q_n x}. \tag{3}$$

On the other hand

$$\left| \frac{1}{n} (e^{2\pi i q_n \alpha} - 1) e^{2\pi i q_n x} \right| = \frac{1}{n} 2 |\sin q_n \pi \alpha| < 2K/n^2.$$

Thus the right-hand side of (3) is uniformly convergent, so denoting it by $\tilde{l}_\alpha(x)$, the function $\tilde{l}_\alpha(x)$ is continuous on $\mathbf{R}$.

Let S be the set of points where the averages of the partial sums of the Fourier series (the Fejér means) of $\tilde{h}$ converge to $\tilde{h}(x)$. According to Lebesgue's theorem S contains the Lebesgue points of $\tilde{h}$, so its complement is a null-set. Changing $\tilde{h}$ on this null-set we can make $\tilde{h}(x)$ to be equal to the limit of the Fejér means in each points where it exists, so we can assume that S is also the set of points where the Fejér means converge.

Since the Fejér means of $\tilde{l}_\alpha(x)$ converge to $\tilde{l}_\alpha(x)$ everywhere, the Fejér means of $\tilde{h}(x)$ and $\tilde{h}(x + \alpha)$ converge simultaneously, thus $x \in S$ if and only if $x + \alpha \in S$. Therefore S is a dense union of translated copies of A. If $x \in S$ then, according to (3), $\Delta_\alpha \tilde{h}(x)$ and $\tilde{l}_\alpha(x)$ are the limits of the averages of the partial sums of the same Fourier series, thus $\Delta_\alpha \tilde{h}(x) = \tilde{l}_\alpha(x)$. Therefore denoting the real part of $\tilde{l}_\alpha$ by l_α we get

$$\Delta_\alpha \overline{h}(x) = l_\alpha(x) \qquad (x \in S,\ \alpha \in A).$$

Now applying Lemma 2, there exists a function $h(x)$ on $\mathbf{R}$ such that $h|S = \overline{h}|S$ and

$$\Delta_\alpha h(x) = l_\alpha(x) \qquad (x \in \mathbf{R},\ \alpha \in A).$$

In particular $\Delta_1 h(x) = l_1(x) = 0$, which implies that h is periodic modulo 1; for any $\alpha \in \Lambda \subset A$ we get $\Delta_\alpha h = l_\alpha$, which shows that $\Delta_\alpha h$ is continuous for any $\alpha \in \Lambda$. Since h and $\overline{h}$ are equal almost everywhere and $\overline{h} \in L_2 \setminus L_\infty$ we get $h \in L_2 \setminus L_\infty$. This completes the proof.

Theorem 2. *For any pseudo-Dirichlet set $\Lambda \subset \mathbf{R}$ (e.g. for any countable Λ and also for some nonempty perfect Λ) there exists a modulo 1 periodic function $h \in L_2 \setminus L_\infty$ and there are modulo α periodic functions $g_\alpha \in L_2 \setminus L_\infty$ for all $\alpha \in \Lambda$ such that $g_\alpha + h$ is continuous for all $\alpha \in \Lambda$.*

Proof. Let $h \in L_2 \setminus L_\infty$ be a modulo 1 periodic function with continuous $\Delta_\alpha h$ for $\alpha \in \Lambda$. Since $\Delta_\alpha h$ is continuous there exists a continuous function f_α for each $\alpha \in \Lambda$ such that $\Delta_\alpha f_\alpha = \Delta_\alpha h$. (After defining f_α arbitrarily in $[0, \alpha]$ such that $f_\alpha(\alpha) - f_\alpha(0) = \Delta_\alpha h(0)$, the equation $f_\alpha(x+\alpha) - f_\alpha(x) = \Delta_\alpha h(x)$ gives a continuous extension.) Then the functions $g_\alpha = f_\alpha - h$ $(\alpha \in \Lambda)$ are periodic modulo α, are in $L_2 \setminus L_\infty$ and the functions $g_\alpha + h = f_\alpha$ are continuous.

Theorem 3. *The sum of two periodic measurable functions can be continuous and unbounded.*

Proof. Using Theorem 2 for $\Lambda = \{\alpha\}$, where α is irrational, we get functions $h, g \in L_2 \setminus L_\infty$ periodic modulo 1 and α resp. such that $f = g + h$ is continuous. But according to Proposition 2, f cannot be bounded.

Corollary. *There exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ that has a measurable $(\alpha_1, \dots, \alpha_n)$-decomposition but does not have a continuous $(\alpha_1, \dots, \alpha_n)$-decomposition, nor has a continuous $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition.*

Proof. The unbounded continuous sum of the two periodic measurable functions constructed above clearly has a measurable (α_1, α_2)-decomposition but does not have a continuous (α_1, α_2)-decomposition (since otherwise it would be bounded). On the other hand, by Proposition 3, this function cannot have an $(\alpha_1, \dots, \alpha_n)$-quasi-decomposition either.

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